Last time:

$$\langle \Psi_0 | T\, \varphi_H(x)\, \varphi_H(y) | \Psi_0 \rangle = \frac{\langle 0 | T\left\{ \varphi_I(x)\varphi_I(y)\, e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} \right\} | 0 \rangle}{\langle 0 | T\, e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} | 0 \rangle}$$

with $H_I = \frac{\lambda}{4!}\int d^4z\, \varphi_I^4(z)$ for $\varphi^4$ theory.

One can expand in powers of $\lambda$ (powers of $H_I$)

and use the fact that

$$\varphi_I(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[ \hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^+_{\vec{k}}\, e^{ik\cdot x} \right] = \varphi_I^+ + \varphi_I^-.$$

to find each term, like $\langle 0 | T\{ \varphi_I(x_1) \ldots \varphi_I(x_n)\} | 0 \rangle$.

## Wick's Theorem (cont'd)

Def. Normal ordering $:\hat{a}\,\hat{a}^+: = \hat{a}^+\hat{a}$ ...

move all $\hat{a}^+$'s to the left of all $\hat{a}$'s. Note: $\langle 0 | : \ldots : | 0 \rangle = 0$ always!

$$:\varphi_I(x)\,\varphi_I(y): = \varphi_I^+(x)\,\varphi_I^+(y) + \varphi_I^-(x)\,\varphi_I^+(y) + \varphi_I^-(y)\,\varphi_I^+(x) + \varphi_I^-(x)\,\varphi_I^-(y).$$

Def. Wick contraction: $\varphi(x)\,\varphi(y) \equiv T\,\varphi(x)\varphi(y) - :\varphi(x)\varphi(y):$

Note that $\langle 0| : \varphi_I(x_1) \dots \varphi_I(x_n) : |0\rangle = 0$. (VEV=0)

<u>Example</u>: $:\hat{a}_{\vec{k}_1}\hat{a}^+_{\vec{k}_2}: = \hat{a}^+_{\vec{k}_2}\hat{a}_{\vec{k}_1}$

$:\hat{a}_{\vec{k}_1}\hat{a}^+_{\vec{k}_2}\hat{a}_{\vec{k}_3}: = \hat{a}^+_{\vec{k}_2}\hat{a}_{\vec{k}_4}\hat{a}_{\vec{k}_3}$

(Def.) <u>Contraction</u> (or Wick contraction) of two fields:

$$\boxed{\overbrace{\varphi(x)\,\varphi}(y) \equiv T\varphi(x)\varphi(y) - :\varphi(x)\varphi(y):}$$

$\Rightarrow$ can see that $\langle 0|\overbrace{\varphi(x)\varphi}(y)|0\rangle = \langle 0|T\varphi(x)\varphi(y)|0\rangle$

as $\langle 0|:\varphi(x)\varphi(0):|0\rangle = 0$.

$\Rightarrow$ contraction is the propagator!

$$T\varphi_I(x)\varphi_I(y) = \theta(x^0-y^0)\varphi_I(x)\varphi_I(y) + \theta(y^0-x^0)\varphi_I(y)\varphi_I(x).$$

$$\Rightarrow T\varphi_I(x)\varphi_I(y) - :\varphi(x)\varphi(y): = \theta(x^0-y^0)\left[\varphi_I^+(x)+\varphi_I^-(x)\right]\cdot$$

$$\cdot\left[\varphi_I^+(y)+\varphi_I^-(y)\right] + \theta(y^0-x^0)\left[\varphi_I^+(y)+\varphi_I^-(y)\right]\left[\varphi_I^+(x)+\varphi_I^-(x)\right]$$

$$-\left(\varphi_I^+(x)\varphi_I^+(y) + \varphi_I^-(x)\varphi_I^+(y) + \varphi_I^-(y)\varphi_I^+(x) + \varphi_I^-(x)\varphi_I^-(y)\right) =$$

$$= \theta(x^0-y^0)\cdot\left[\varphi_I^+(x),\varphi_I^-(y)\right] + \theta(y^0-x^0)\left[\varphi_I^+(y),\varphi_I^-(x)\right]$$

Hence

$$\overbracket{\varphi(x)\,\varphi(y)} = \theta(x^0-y^0)\left[\varphi_I^+(x),\, \varphi_I^-(y)\right] + \theta(y^0-x^0)\left[\varphi_I^+(y),\, \varphi_I^-(x)\right]$$

$$\left[\varphi_I^+(x),\, \varphi_I^-(y)\right] = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\, \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}}\, e^{-ik\cdot x + ik'\cdot y} \underbrace{\left[\hat{a}_{\vec{k}},\, \hat{a}^+_{\vec{k}'}\right]}_{(2\pi)^3 2\varepsilon_k \delta(\vec{k}-\vec{k}')}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\, e^{-ik\cdot(x-y)} = D(x-y) \left( = \langle 0|\, \varphi(x)\varphi(y)\, |0\rangle \right).$$

$$\Rightarrow \overbracket{\varphi(x)\,\varphi(y)} = \theta(x^0-y^0)\, D(x-y) + \theta(y^0-x^0)\, D(y-x)$$

$$= D_F(x-y) \qquad \text{(just a function!)}$$

$$\Rightarrow \boxed{\overbracket{\varphi(x)\,\varphi(y)} = D_F(x-y)} \;\sim\; \text{Feynman propagator!}$$

**Wick's theorem** $T\, \varphi(x_1)\varphi(x_2)\dots\varphi(x_n) = :\varphi(x_1)\varphi(x_2)\dots$

(all $\varphi$'s are in the interaction picture)

$\dots\varphi(x_n): + :\overbracket{\varphi(x_1)\,\varphi(x_2)}\,\varphi(x_3)\dots\varphi(x_n): +$ (all possible 1-contractions)

$+ :\overbracket{\varphi(x_1)\,\varphi(x_2)}\,\overbracket{\varphi(x_3)\,\varphi(x_4)}\,\varphi(x_5)\dots\varphi(x_n): +$ (all possible 2-contractions)

$+ \dots + \underbrace{\overbracket{\varphi(x_1)\varphi(x_2)}\,\overbracket{\varphi(x_3)\varphi(x_4)}\dots\overbracket{\varphi(x_{n-1})\,\varphi(x_n)} + \dots}_{\text{only non-normal-ordered terms}}$ (for even $n$)

*Example:* $T\,\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) = :\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4):$

$+ :\overbracket{\varphi(x_1)\,\varphi(x_2)}\,\varphi(x_3)\,\varphi(x_4): + :\overbracket{\varphi_1\,\varphi_2\,\varphi_3}\,\varphi_4: + :\overbracket{\varphi_1\,\varphi_2\,\varphi_3\,\varphi_4}:$

$+ :\varphi_1\overbracket{\varphi_2\,\varphi_3}\varphi_4: + :\varphi_1\,\overbracket{\varphi_2\,\varphi_3\,\varphi_4}: + :\varphi_1\,\varphi_2\,\overbracket{\varphi_3\,\varphi_4}: + \overbracket{\varphi_1\,\varphi_2}\,\overbracket{\varphi_3\,\varphi_4} +$

$$+\ \wick{\c1{\varphi_1}\c2{\varphi_2}\c1{\varphi_3}\c2{\varphi_4}} \quad + \quad \wick{\c1{\varphi_1}\c2{\varphi_2}\c2{\varphi_3}\c1{\varphi_4}}\ .$$

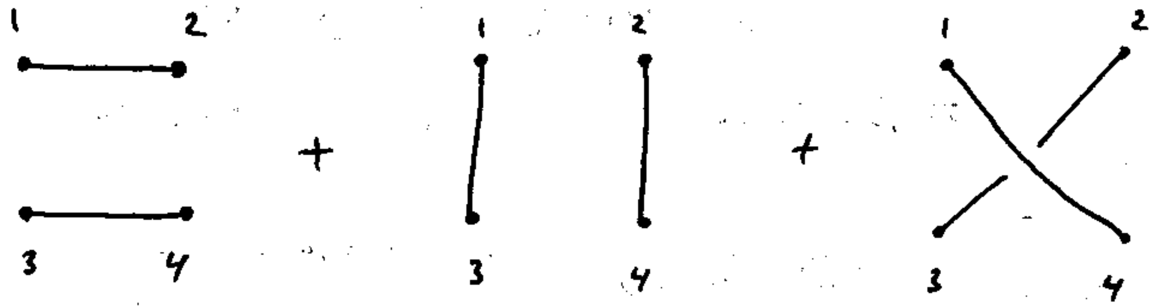
Hence

$$\langle 0|T\,\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle = \wick{\c{\varphi_1}\c{\varphi_2}\c{\varphi_3}\c{\varphi_4}} +$$

$$+\ \wick{\c1{\varphi_1}\c2{\varphi_2}\c1{\varphi_3}\c2{\varphi_4}} \quad + \quad \wick{\c1{\varphi_1}\c2{\varphi_2}\c2{\varphi_3}\c1{\varphi_4}} = D_F(x_1-x_2)\,D_F(x_3-x_4)$$

$$+\ D_F(x_1-x_3)\,D_F(x_2-x_4) + D_F(x_1-x_4)\,D_F(x_2-x_3).$$

Diagrammatically:




~ simple example of Feynman diagrams

<u>Proof of Wick's theorem</u>: by induction.

$\Rightarrow$ True for $n=2$ (aka definition of contraction).

$\Rightarrow$ assume it is true for $n-1$

$\Rightarrow$ assume that $x_1^0 \geqslant x_2^0 \geqslant \dots \geqslant x_n^0 \Rightarrow$

$$T(\varphi_1\varphi_2\dots\varphi_n) = \varphi_1\varphi_2\dots\varphi_n = \varphi_1\cdot T(\varphi_2\dots\varphi_n) =$$

$$= (\text{apply the th'm for } n-1) = \varphi_1\cdot\left[:\varphi_2\dots\varphi_n: + \text{contractions}\right]$$

$$= \Big(\underset{\hat{a}}{\varphi_1^+} + \underset{\hat{a}^+}{\varphi_1^-}\Big)\left[:\varphi_2\dots\varphi_n: + \text{contractions}\right] =$$

$= \varphi_1^+ \left[ :\varphi_2 \dots \varphi_n: + \text{contractions} \right] + :\varphi_1^- \varphi_2 \dots \varphi_n: +$

$+$ all contractions without $\varphi_1$ $\qquad (*)$

as $\varphi_1^- \propto \hat{a}^+$ ~ already normal-ordered.

What about $\varphi_1^+$? Consider one term:

$$\varphi_1^+ :\varphi_2 \dots \varphi_n: = :\varphi_2 \dots \varphi_n: \varphi_1^+ + \left[ \varphi_1^+, :\varphi_2 \dots \varphi_n: \right]$$

$$= :\varphi_1^+ \varphi_2 \dots \varphi_n: + :\left[\varphi_1^+, \underbrace{\varphi_2}_{=\cancel{\varphi_2^+} + \varphi_2^-}\right] \varphi_3 \dots \varphi_n: + :\varphi_2 \left[\varphi_1^+, \underbrace{\varphi_3}_{=\cancel{\varphi_3^+}+\varphi_3^-}\right] \varphi_4 \dots \varphi_n: + \dots$$

$$= :\varphi_1^+ \varphi_2 \dots \varphi_n: + :\underbrace{\left[\varphi_1^+, \varphi_2^-\right]}_{\varphi_1 \varphi_2} \varphi_3 \dots \varphi_n: + :\varphi_2 \underbrace{\left[\varphi_1^+, \varphi_3^-\right]}_{\varphi_1 \varphi_3} \varphi_4 \dots \varphi_n: + \dots =$$

$$= \underbrace{:\varphi_1^+ \varphi_2 \dots \varphi_n:} + \text{all 1-contractions including } \varphi_1$$

$+ :\varphi_1^- \varphi_2 \dots \varphi_n: = :\varphi_1 \varphi_2 \dots \varphi_n:$

$\Rightarrow$ repeat the same for other terms in $(*)$ $\Rightarrow$ prove the theorem.

---

## Feynman Diagrams in $\varphi^4$ Theory.

Using Wick's theorem let us evaluate the 2-point function:

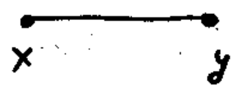
$$\langle \Psi_0 | T \varphi(x) \varphi(y) | \Psi_0 \rangle = \frac{1}{\langle 0 | T \exp\left\{ -i \int d^4x \frac{\lambda}{4!} \varphi_I^4(x) \right\} | 0 \rangle} \cdot$$

$$\left\{ \langle 0 | T \varphi_I(x) \varphi_I(y) | 0 \rangle - i \frac{\lambda}{4!} \int d^4 z \, \langle 0 | T \left\{ \varphi_I(x) \varphi_I(y) \varphi_I^4(z) \right\} | 0 \rangle \right.$$

$$- \frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 \int d^4z_1\, d^4z_2 \ \langle 0 | T \left\{ \varphi_I(x)\varphi_I(y)\, \varphi_I^4(z_1)\, \varphi_I^4(z_2) \right\} | 0 \rangle + \dots \Big\}$$

Start with the numerator:

$O(\lambda^0)$: $\langle 0 | T\, \varphi_I(x)\, \varphi_I(y) | 0 \rangle$ =




$O(\lambda)$:

$$-i\frac{\lambda}{4!}\int d^4z \ \langle 0 | T\left\{\varphi_I(x)\,\varphi_I(y)\,\varphi_I^4(z)\right\} | 0 \rangle \overset{\text{Wick's th'm}}{=} \text{sum over all contractions}$$

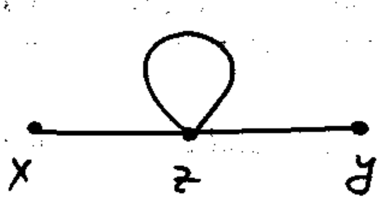
$$= -i\frac{\lambda}{4!}\int d^4z \Big\{ 4\cdot 3\, \overbrace{\varphi_I(x)\ \varphi_I(z)}\ \overbrace{\varphi_I(y)\ \varphi_I(z)}\ \overbrace{\varphi_I(z)\ \varphi_I(z)}$$

$$+ 3\cdot \overbrace{\varphi_I(x)\ \varphi_I(y)}\ \overbrace{\varphi_I(z)\ \varphi_I(z)}\ \overbrace{\varphi_I(z)\ \varphi_I(z)} \Big\} =$$

$$= -i\frac{\lambda}{4!}\int d^4z \Big\{ 12\, D_F(x-z)\, D_F(y-z)\, D_F(z-z)$$

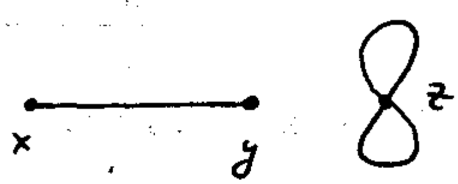
$$+ 3\, D_F(x-y)\, D_F(z-z)\, D_F(z-z) \Big\}$$

$\underbrace{\frac{12}{4!}}_{\text{symmetry factor}} = \frac{1}{2}$



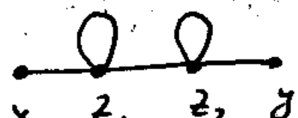

"tadpole diagram"

$\frac{3}{4!} = \frac{1}{8}$

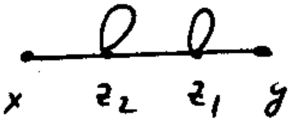



"disconnected diagram"

$O(\lambda^2)$: $-\frac{\lambda^2}{2(4!)^2}\Big\{ 16\cdot 9$

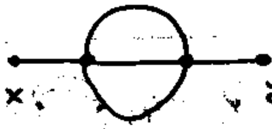
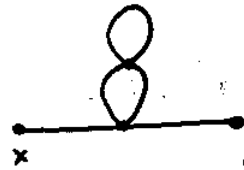



$+ 16\cdot 9$

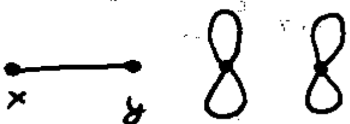



$+$

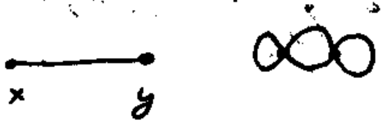
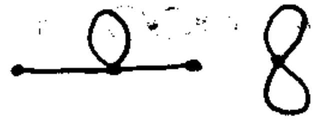
identical ⇒ just double

$+$  $4\cdot4\cdot3!\cdot2$ $+$  $4\cdot3\cdot4\cdot3\cdot2$

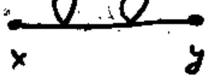
"sunset diagram" $\quad z_1 \leftrightarrow z_2 \quad$ "cactus diagram" $\quad z_1 \leftrightarrow z_2$

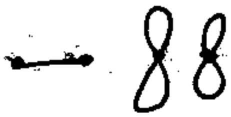
$+ \; 9$  $+ \; 4!$  $+$

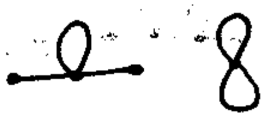
"basketball diagram"

$+ \; 6^2\cdot 2$  $+ \; 12\cdot3\cdot2$  $\Big\} =$

$z_1 \leftrightarrow z_2$

$= -\lambda^2 \Big\{ \frac{1}{4}$  $+ \frac{1}{6}$  $+ \frac{1}{4}$  $+$

$+ \frac{1}{128}$  $+ \frac{1}{48}$  $+ \frac{1}{16}$ 

$+ \frac{1}{16}$  $\Big\}$

$\Rightarrow$ Inverse coefficients are called <u>symmetry factors</u>.

$\Rightarrow$ What about the denominator?

$$\langle 0 | T e^{-i\frac{\lambda}{4!}\int d^4z\, \varphi_I^4(z)} | 0\rangle = 1 - i\frac{\lambda}{4!}\int d^4z \langle 0 | \varphi_I^4(z) | 0\rangle -$$

$$- \frac{1}{2!}\frac{\lambda^2}{(4!)^2}\int d^4z_1\, d^4z_2 \langle 0 | T\, \varphi_I^4(z_1)\, \varphi_I^4(z_2) | 0\rangle + \dots =$$

$= 1 - i\,\frac{\lambda}{4!}\; 3$  $- \frac{\lambda^2}{2!(4!)^2}\Big[\, 9$  $+ \; 4!$  $+ \; 72$  $\Big] + \dots$

$\Rightarrow$ combining the numerator & denominator we write