<u>Last time</u>: showed that $\overbracket{\varphi_I(x)\,\varphi_I(y)} = D_F(x-y)$.

Proved <u>Wick's theorem</u>:

$$T\,\varphi(x_1)\,\varphi(x_2)\dots\varphi(x_n) = :\varphi_1\varphi_2\dots\varphi_n: + :\overbracket{\varphi_1\varphi_2}\,\varphi_3\dots\varphi_n: +$$

$$+ (\text{all 1-contractions}) + \dots + :\overbracket{\varphi_1\varphi_2}\,\overbracket{\varphi_3\varphi_4}\dots\overbracket{\varphi_{n-1}\varphi_n}: +$$

$$+ \left(\text{all } \tfrac{n}{2}\text{-contractions}\right) \quad \text{if } n \text{ is even.}$$

For odd n the last term is $:\overbracket{\varphi_1\varphi_2}\dots\overbracket{\varphi_{n-2}\varphi_{n-1}}\,\varphi_n: +$

$$+\left(\text{all } \tfrac{n-1}{2}\text{-contractions}\right).$$

<u>Main consequence</u>: for even n get

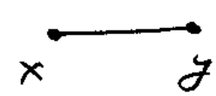
$$\langle 0|\,T\,\varphi_I(x_1)\dots\varphi_I(x_n)\,|0\rangle = \overbracket{\varphi_1\varphi_2}\dots\overbracket{\varphi_{n-1}\varphi_n} + \text{permutations}$$

For $n=4$ get 

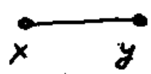
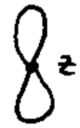
<u>Feynman Diagrams in φ^4 Theory</u> (cont'd).

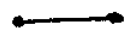
$$\langle \Psi_0|\,T\,\varphi(x)\varphi(y)\,|\Psi_0\rangle = \frac{\langle 0|\,T\left\{\varphi_I(x)\varphi_I(y)\,e^{-i\int d^4z\,\frac{\lambda}{4!}\varphi_I^4(z)}\right\}|0\rangle}{\langle 0|\,T\,e^{-i\int d^4z\,\frac{\lambda}{4!}\varphi_I^4(z)}\,|0\rangle}$$

Expanding the exponent in the numerator we got:

$O(\lambda^0)$: $\langle 0|\,T\,\varphi_I(x)\varphi_I(y)\,|0\rangle =$ 

$O(\lambda)$: $\; -i \dfrac{\lambda}{4!} \displaystyle\int d^4 z \, \langle 0 | T \, \varphi_I(x) \, \varphi_I(y) \, \varphi_I^{\,4}(z) | 0 \rangle \; =$

$= -i \dfrac{\lambda}{4!} \Big[\, 12$  $+ \; 3$   $\Big] \; =$

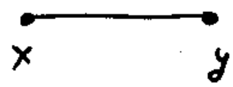
$= -i \lambda \Big[\, \dfrac{1}{2}$  $+ \; \dfrac{1}{8}$   $\Big]$.

The one over numerical prefactor's are called symmetry factors. We got $S = 2$ and $S = 8$.

(Symmetry factors are labeled S.)

$$- \frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 \int d^4z_1\, d^4z_2 \, \langle 0 | T \{ \varphi_I(x)\varphi_I(y)\varphi_I^4(z_1)\varphi_I^4(z_2)\} |0\rangle + \ldots \Big\}$$

Start with the numerator:

$O(\lambda^0)$: $\langle 0 | T \varphi_I(x) \varphi_I(y) |0\rangle =$ 

$O(\lambda)$:

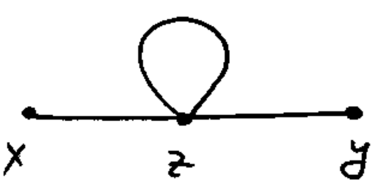
$$-i\frac{\lambda}{4!}\int d^4z \, \langle 0 | T\{\varphi_I(x)\varphi_I(y)\varphi_I^4(z)\}|0\rangle \overset{\text{Wick's th'm}}{=} \text{sum over all contractions}$$

$$= -i\frac{\lambda}{4!}\int d^4z \Big\{ 4\cdot 3\, \overline{\varphi_I(x)\varphi_I(z)}\;\overline{\varphi_I(y)\varphi_I(z)}\;\overline{\varphi_I(z)\varphi_I(z)}$$

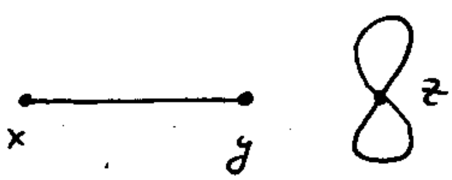
$$+ 3\cdot \overline{\varphi_I(x)\varphi_I(y)}\;\overline{\varphi_I(z)\varphi_I(z)}\;\overline{\varphi_I(z)\varphi_I(z)} \Big\} =$$

$$= -i\frac{\lambda}{4!}\int d^4z \Big\{ 12\, D_F(x-z)\, D_F(y-z)\, D_F(z-z)$$

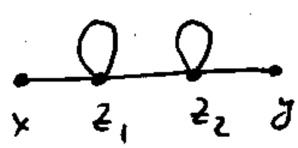
$$+ 3\, D_F(x-y)\, D_F(z-z)\, D_F(z-z) \Big\}$$

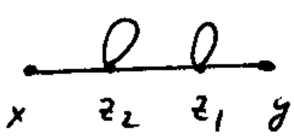
$\underbrace{\frac{12}{4!}}_{\text{symmetry factor}} = \frac{1}{2}$ 

"tadpole diagram"

$\frac{3}{4!} = \frac{1}{8}$ 

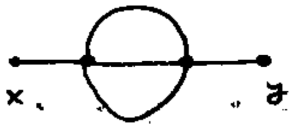
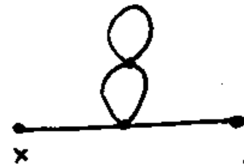
"disconnected diagram"

$O(\lambda^2)$: $-\frac{\lambda^2}{2(4!)^2}\Big\{ 16\cdot 9$ 

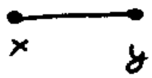
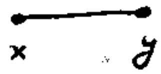
$+ 16\cdot 9$ 

$+$

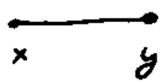
identical ⇒ just double

$+$  $4 \cdot 4 \cdot 3! \cdot 2$ $+$  $4 \cdot 3 \cdot 4 \cdot 3 \cdot 2$

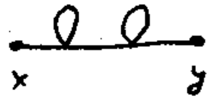
"sunset diagram" $\quad z_1 \leftrightarrow z_2 \quad$ "cactus diagram" $\quad z_1 \leftrightarrow z_2$

$+ \; 9$   $+ \; 4!$   $+$

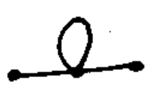
"basketball diagram"

$+ \; 6^2 \cdot 2$   $+ \; 12 \cdot 3 \cdot 2$   $\Big\} =$

$z_1 \leftrightarrow z_2$

$= -\lambda^2 \Big\{ \frac{1}{4}$  $+ \frac{1}{6}$  $+ \frac{1}{4}$  $+$

$+ \frac{1}{128}$   $+ \frac{1}{48}$   $+ \frac{1}{16}$  

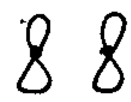
$+ \frac{1}{16}$   $\Big\}$

$\Rightarrow$ Inverse Coefficients are called <u>symmetry factors.</u>

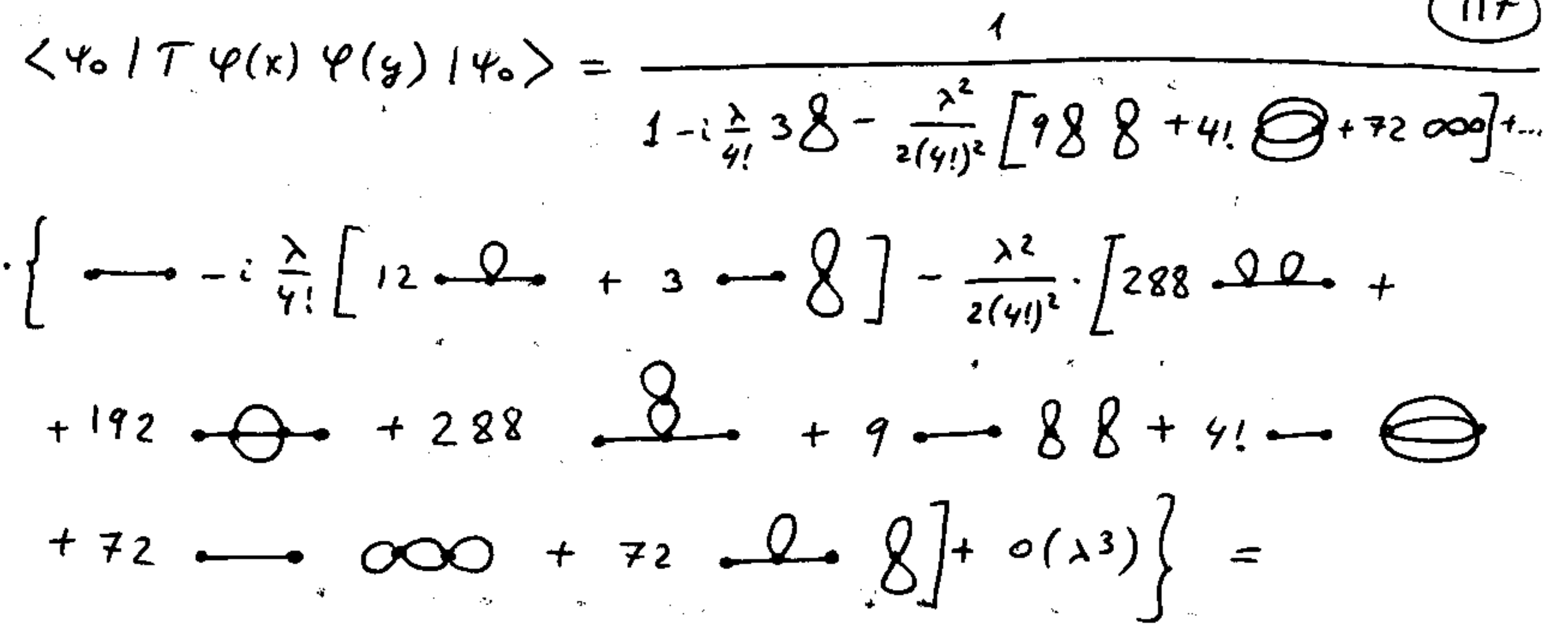
$\Rightarrow$ What about the denominator?

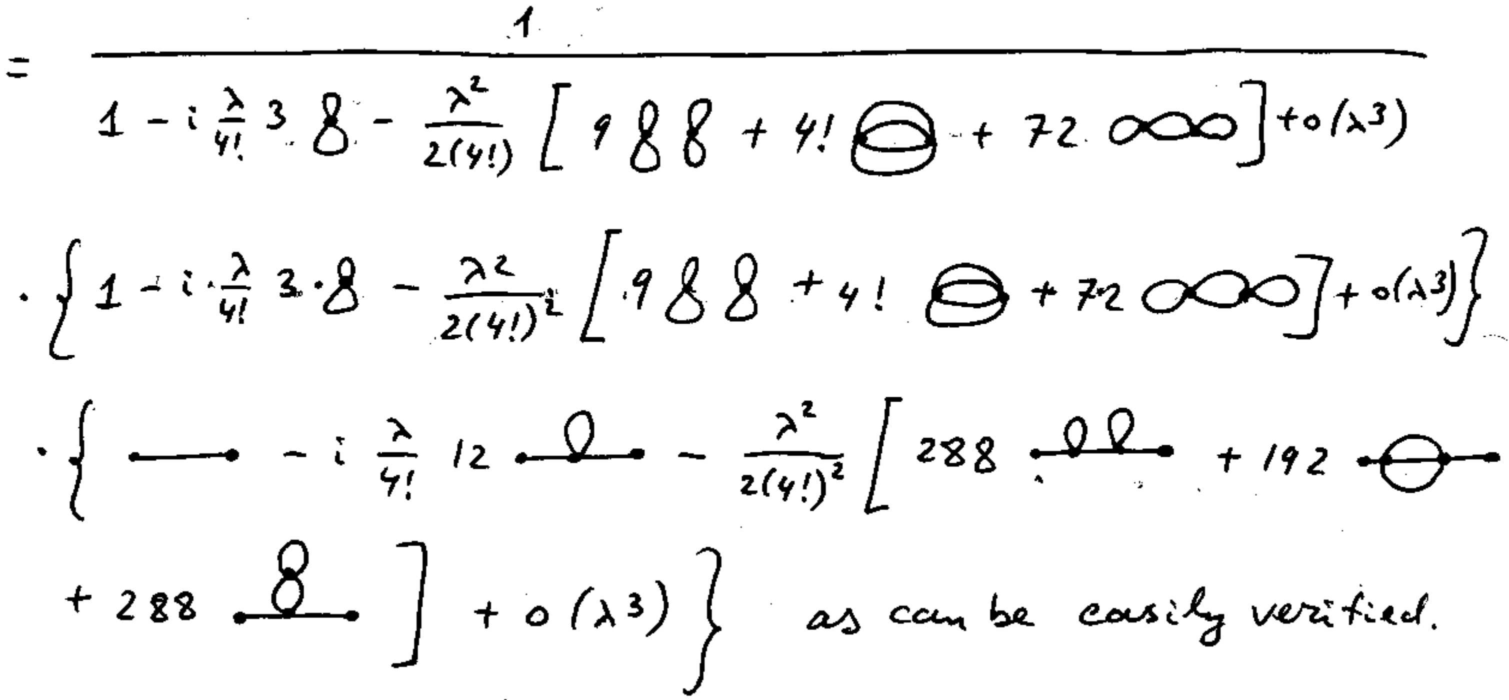
$$\langle 0 | T e^{-i \frac{\lambda}{4!} \int d^4 z \, \varphi_I^4(z)} | 0 \rangle = 1 - i \frac{\lambda}{4!} \int d^4 z \langle 0 | \varphi_I^4(z) | 0 \rangle -$$

$$- \frac{1}{2!} \frac{\lambda^2}{(4!)^2} \int d^4 z_1 \, d^4 z_2 \, \langle 0 | T \varphi_I^4(z_1) \varphi_I^4(z_2) | 0 \rangle + \ldots =$$

$= 1 - i \frac{\lambda}{4!} \; 3$  $- \frac{\lambda^2}{2!(4!)^2} \Big[9$  $+ 4!$  $+ 72$  $\Big] + \ldots$

$\Rightarrow$ combining the numerator & denominator we write

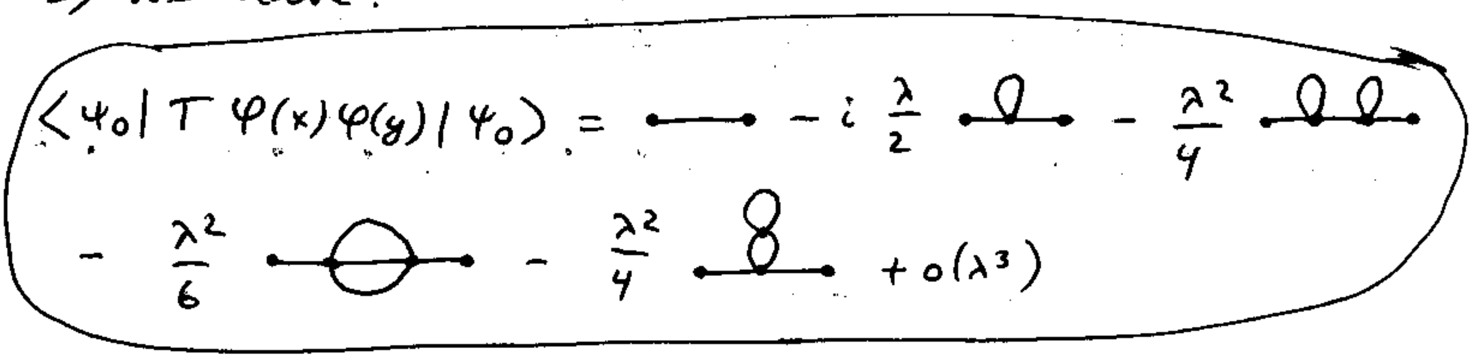




as can be easily verified.

$\Rightarrow$ We see that the denominator simply cancels disconnected diagrams!

$\Rightarrow$ We have:



$\Rightarrow$ can prove cancellation of disconnected graphs in general:

Note that if we look at the whole expression:

$$\langle \Psi_0 | T\, \varphi_H(x)\varphi_H(y) | \Psi_0 \rangle = \frac{\langle 0 | T\{ \varphi_I(x)\varphi_I(y)\, e^{-i\int d^4z \frac{\lambda}{4!}\varphi_I^4(z)} \} | 0 \rangle}{\langle 0 | T e^{-i\int d^4z \frac{\lambda}{4!}\varphi_I^4(z)} | 0 \rangle}$$

we can see that <u>disconnected</u> diagrams arize from self-contractions between $\varphi_I(z_i)$ (i labels different z-integrals arising from expanding the exponential).

We then write:

$$\text{Numerator} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \langle 0 | T\{ \varphi_I(x)\varphi_I(y)\, \varphi_I^4(z_1) \ldots \varphi_I^4(z_n) \} | 0 \rangle$$

$$\cdot\, d^4z_1 \ldots d^4z_n \left(-i\frac{\lambda}{4!}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} \left(-i\frac{\lambda}{4!}\right)^n \int d^4z_1 \ldots d^4z_n \cdot$$

$$\cdot \langle 0 | T\{ \varphi_I(x)\varphi_I(y)\, \varphi_I^4(z_1) \ldots \varphi_I^4(z_m) \} | 0 \rangle_{\substack{\text{connected}\\ \text{only}}} \cdot \langle 0 | T\, \varphi_I(z_{m+1}) \ldots$$

$$\ldots \varphi_I(z_n) | 0 \rangle = \Big| k = n - m = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left(-i\frac{\lambda}{4!}\right)^{k+m} \frac{1}{k!\, m!}$$

$$\cdot \int d^4z_1 \ldots d^4z_m\, d^4z_{m+1} \ldots d^4z_{m+k}\, \langle 0 | T\{ \varphi_I(x)\varphi_I(y)\, \varphi_I^4(z_1) \ldots$$

$$\ldots \varphi_I^4(z_m) \} | 0 \rangle_{\text{conn.}} \cdot \langle 0 | T\, \varphi_I(z_{m+1}) \ldots \varphi_I(z_{m+k}) | 0 \rangle =$$

$$= \langle 0 | T\{ \varphi_I(x)\varphi_I(y)\, e^{-i\frac{\lambda}{4!}\int d^4z\, \varphi_I^4(z)} \} | 0 \rangle_{\text{conn}} \cdot \langle 0 | T e^{-\frac{i}{4!}\lambda\int d^4z\, \varphi_I^4(z)} | 0 \rangle$$

$$\Rightarrow \boxed{\begin{aligned}&\langle \Psi_0 | T\, \varphi_H(x)\varphi_H(y)|0\rangle = \\ &= \langle 0 | T\left\{ \varphi_I(x)\varphi_I(y)\, e^{-i\frac{\lambda}{4!}\int d^4z\, \varphi_I^4(z)} \right\}|0\rangle_{\text{conn.}}\end{aligned}}$$

as desired.

$\Rightarrow$ ibid for other correlators of higher order!

$\Rightarrow$ Note that disconnected graphs exponentiate:

above we had

Denominator $= \exp\Big\{ -i\frac{\lambda}{4!}\,3$  $- \frac{\lambda^2}{2\cdot 4!}$  $- \frac{\lambda^2}{16}$  $+ \ldots \Big\}.$

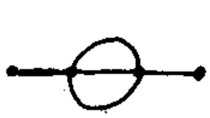
(see Peskin pp. 96–98.)

$\Rightarrow$ We are now ready to formulate the rules for Feynman diagrams calculation.

$\Rightarrow$ more on calculation of symmetry factors:

(see attached handout)

 $S_1 = 2!\cdot 2\cdot 2 = 8\,,\quad S_2 = 2 \Rightarrow \frac{1}{S} = \frac{1}{16}$ ok.

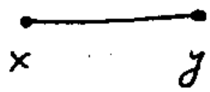
 $S_1 = 3! = 6\,,\quad S_2 = 1 \Rightarrow \frac{1}{S} = \frac{1}{6}\,.$

 $S_1 = 2^4\,(2!)^2 = 64$, $S_2 = 2 \Rightarrow \frac{1}{S} = \frac{1}{128}$

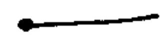
 $S_1 = 4$, $S_2 = 1 \Rightarrow \frac{1}{S} = \frac{1}{4}$

Feynman rules for φ^4-theory in coordinate space:

(for correlation functions)

① Each propagator gives  $= D_F(x-y)$

② Each vertex gives  $= -i\lambda \int d^4 z$

③ Each external point  $= 1$.

④ Divide by symmetry factors.

⑤ Keep connected diagrams only.

Often it is important to find observables in momentum space. It is also easier to calculate Feynman diagrams in momentum space.

Def. n-point "Green function":

Take $\boxed{G(x_1, x_2, \ldots, x_n) = \langle \Psi_0 | T\{\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\}|\Psi_0\rangle}$.

In momentum space write:

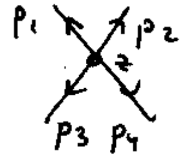
$$\tilde{G}(p_1, p_2, \ldots, p_n) = \int d^4x_1\, d^4x_2 \ldots d^4x_n\, e^{ip_1\cdot x_1 + ip_2\cdot x_2 + \ldots + ip_n\cdot x_n}$$

$G(x_1, x_2, \ldots, x_n)$ is the "Green function" in momentum space.

$\Rightarrow$ Each propagator $D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} e^{-ik\cdot(x-y)}$

gives $\frac{i}{k^2-m^2+i\varepsilon}$ in momentum space.

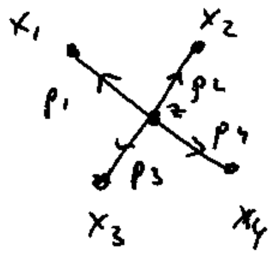
$\Rightarrow$ Each vertex gives: $-i\lambda \int d^4z \; e^{ip_1\cdot z + ip_2 z + ip_3 z + ip_4 z} =$



$= -i\lambda\, (2\pi)^4\, \delta^{(4)}(p_1+p_2+p_3+p_4)$

$\sim$ conservation of energy & momentum.

<u>Example</u>



$= \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4 \; e^{ip_1x_1 + ip_2x_2 + ip_3x_3 + ip_4x_4} \cdot (-i\lambda)\int d^4z \; D_F(x_1-z)\, D_F(x_2-z)\, D_F(x_3-z) \cdot$

$\cdot D_F(x_4-z) = \left|\; \tilde{x}_i = x_i - z \right. = \int d^4\tilde{x}_1\, d^4\tilde{x}_2\, d^4\tilde{x}_3\, d^4\tilde{x}_4 \cdot$

$\cdot e^{ip_1\tilde{x}_1 + ip_2\tilde{x}_2 + ip_3\tilde{x}_3 + ip_4\tilde{x}_4} \; D_F(\tilde{x}_1)\, D_F(\tilde{x}_2)\, D_F(\tilde{x}_3)\, D_F(\tilde{x}_4) \cdot$

$(-i\lambda)\int d^4z \; e^{ip_1z_1 + ip_2z_2 + ip_3z_3 + ip_4z_4} = \frac{i}{p_1^2-m^2+i\varepsilon} \cdot$

$\cdot \frac{i}{p_2^2-m^2+i\varepsilon} \; \frac{i}{p_3^2-m^2+i\varepsilon} \; \frac{i}{p_4^2-m^2+i\varepsilon} \; (-i\lambda)\, \underbrace{(2\pi)^4\, \delta^{(4)}(p_1+p_2+p_3+p_4)}_{\text{overall factor of energy-momentum conservation (usually dropped)} \sim \text{see later}}$