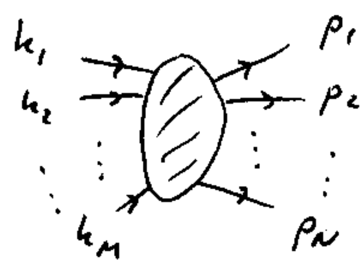
Last time: formulated Feynman rules for Green fn's in momentum space:

1. Each line gives $\xrightarrow{k} = \dfrac{i}{k^2 - m^2 + i\varepsilon}$
2. Each vertex gives $\times = -i\lambda$
3. 4-momentum is conserved at vertices. Integrate over each independent momentum $\dfrac{d^4k}{(2\pi)^4}$.
4. Symmetry factors.
5. Connected graphs only.

Cross Section, S-matrix and Reduction Formulas (cont'd)

Cross Section and S-matrix (cont'd)



Def. S-matrix: $|\psi_f\rangle = S\,|\psi_i\rangle$.

$$S S^\dagger = \mathbb{1}$$

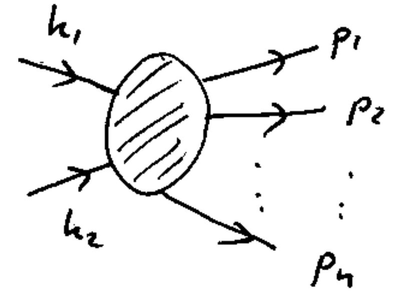
$$S = U(+\infty, -\infty)$$ (interaction picture)

Def. T-matrix: $S = \mathbb{1} + i\,\mathbb{T}$ $\Rightarrow$ $-i\left(T - T^\dagger\right) = T\,T^\dagger$.

Def. Scattering amplitude M:

$$\langle \{p_i\} |\, i\,T\, | \{k_j\} \rangle = (2\pi)^4\, \delta^{(4)}\!\left(\sum_{i=1}^{N} p_i - \sum_{j=1}^{M} k_j\right) i\, M(\{k_j\} \to \{p_i\})$$

$2 \to n$ scattering



$$|\psi_i^2\rangle = \int \frac{d^3q_1\, d^3q_2}{(2\pi)^3 2\mathcal{E}_{q_1} (2\pi)^3 2\mathcal{E}_{q_2}} f_1(q_1) f_2(q_2) |q_1, q_2\rangle$$

$f_1, f_2 \sim$ peaked (sharply) around k_1, k_2.

$$P_{2\to n} = \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2\mathcal{E}_{p_i}} \left|\langle p_1, \ldots, p_n | iT | \psi_i^2\rangle\right|^2.$$

Def. Cross section $(\sigma) = \dfrac{\text{event probability per unit volume \& time}}{(\text{target density}) \times (\text{incident flux of particles})}$.

After much algebra showed that event probability per unit volume & time is:

$$\boxed{\frac{dP_{2\to n}}{d^4x} = |\tilde{f}_1(x)|^2 |\tilde{f}_2(x)|^2 \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2\mathcal{E}_{p_i}} \left|\mathcal{M}(k_1, k_2 \to \{p_i\})\right|^2 \cdot (2\pi)^4 \delta^{(4)}\Big(k_1 + k_2 - \sum_{j=1}^{n} p_j\Big).}$$

where $\tilde{f}_i(x) = \displaystyle\int \frac{d^3q}{(2\pi)^3 2\mathcal{E}_q} e^{-iq\cdot x} f_i(q)$, $i = 1, 2$.

Event probability per unit volume and time is

$$\boxed{\frac{dP_{2\to n}}{d^4x} = |\tilde{f}_1(x)|^2 \, |\tilde{f}_2(x)|^2 \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2\mathcal{E}_i} |M(k_1,k_2\to\{p_j\})|^2 \cdot (2\pi)^4 \delta^{(4)}\left(k_1+k_2-\sum_{j=1}^{n} p_j\right)}$$

Suppose particle k_2 is at rest (target) and k_1 is incident on it. The target density is found by noting that the total # of particles in state

$$|\psi_2\rangle = \int \frac{d^3q}{(2\pi)^3 2\mathcal{E}_q} f_2(q)\,|q\rangle$$

(we can factor $|q_1,q_2\rangle = |q_1\rangle\,|q_2\rangle$ as they are far apart initially).

is

$$N_2 = \frac{\langle\psi_2|\hat{N}|\psi_2\rangle}{\underbrace{\langle\psi_2|\psi_2\rangle}_{=1}} = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \langle\psi_2|\,\hat{a}^+_{\vec{k}}\,\hat{a}_{\vec{k}}\,|\psi_2\rangle =$$

$$= \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \int \frac{d^3q}{(2\pi)^3 2\mathcal{E}_q} \frac{d^3q'}{(2\pi)^3 2\mathcal{E}_{q'}} f_2(q)\, f_2^*(q')\, \langle 0|\underbrace{\hat{a}_{\vec{q}'}\,\hat{a}^+_{\vec{k}}}_{(2\pi)^3 2\mathcal{E}_k \delta(\vec{q}'-\vec{k})}\hat{a}_{\vec{k}} \cdot$$

$$\cdot\, \hat{a}^+_{\vec{q}}\,|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \frac{d^3q}{(2\pi)^3 2\mathcal{E}_q} \frac{d^3q'}{(2\pi)^3 2\mathcal{E}_{q'}} f_2(q)\, f_2^*(q') \cdot$$

$$\cdot (2\pi)^3 2\mathcal{E}_{q'}\, \delta(\vec{q}'-\vec{k})\, (2\pi)^3 2\mathcal{E}_q\, \delta(\underbrace{\vec{q}-\vec{k}}_{}) = \int \frac{d^3q}{(2\pi)^3 2\mathcal{E}_q} \frac{d^3q'}{(2\pi)^3 2\mathcal{E}_{q'}} \cdot$$

$\mathcal{E}_{q'} = k_2$ as $f_2(q)$ is peaked at k_2; $\vec{k} = \vec{q}$

$$\cdot (2\pi)^3 2\mathcal{E}_{k_2}\, \delta(\vec{q}-\vec{q}')\, f_2(q)\, f_2^*(q') = \int d^3x \cdot 2\mathcal{E}_{k_2} |\tilde{f}_2(x)|^2$$

$$\delta(\vec{q}-\vec{q}') = \int \frac{d^3x}{(2\pi)^3} e^{i\vec{x}\cdot(\vec{q}-\vec{q}')} = e^{-ix\cdot(q-q')}$$

The # of particles per unit volume is

$$\boxed{\frac{dN_2}{d^3x} = 2\,\mathcal{E}_{k_2}\,|\tilde{f}_2(x)|^2}.$$

Similarly the incoming particle density is

$$\boxed{\frac{dN_1}{d^3x} = 2\,\mathcal{E}_{k_1}\,|\tilde{f}_1(x)|^2.}$$

If particle k_2 is at rest $\Rightarrow$ $\mathcal{E}_{k_2} = m_2$. The incoming flux is $\frac{dN_1}{d^3x}\cdot v_1$ where $v_1 = \frac{|\vec{k}_1|}{\mathcal{E}_{k_1}}$ is velocity of particle k_1. In the rest frame of target have

$$(\text{target density}) \times (\text{incident flux}) = |\tilde{f}_1(x)|^2\,|\tilde{f}_2(x)|^2 \cdot$$

$$\cdot\, 2m_2 \cdot 2\,\mathcal{E}_{k_1} \cdot \frac{|\vec{k}_1|}{\mathcal{E}_{k_1}} = |\tilde{f}_1(x)|^2\,|\tilde{f}_2(x)|^2\;4\,m_2\,|\vec{k}_1|.$$

In a general frame one can show that

$m_2|\vec{k}_1|$ becomes

$$\boxed{\sqrt{(k_1\cdot k_2)^2 - m_1^2\cdot m_2^2} = \mathcal{E}_{k_1}\,\mathcal{E}_{k_2}\,|\vec{v}_1 - \vec{v}_2|}$$

↖ Lorentz-invariant! ↖ simple to use!

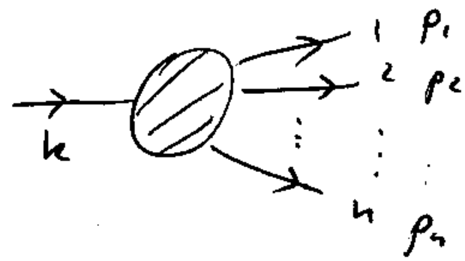
Combining these results with the expression for cross section & for probability density we get

$$d\sigma = \frac{1}{2\mathcal{E}_{k_1}\, 2\mathcal{E}_{k_2}\, |\vec{v}_1 - \vec{v}_2|} \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2\mathcal{E}_i}\, |M(k_1, k_2; p_1, \dots, p_n)|^2 \cdot (2\pi)^4\, \delta^{(4)}\Big(k_1 + k_2 - \sum_{j=1}^{n} p_j\Big)$$

This is our master-formula for cross-sections!

Decay Rate

Now imagine 1 particle decaying into n particles:



Def. Decay rate Γ $= \dfrac{\#\text{ events per unit volume \& time}}{\text{density of particles}}$

$$= \frac{\#\text{ events per unit time}}{\#\text{ particles.}}$$

Calculation is analogous to the above. We now have one particle in initial state: in its rest frame

$2\mathcal{E}_{k_1} \cdot 2\mathcal{E}_{k_2} |\vec{v}_1 - \vec{v}_2| \to 2m$ (m ~ mass of initial particle)

$$\Rightarrow \boxed{d\Gamma = \frac{1}{2m} \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2\mathcal{E}_i}\, |M(k; p_1, \dots, p_n)|^2 \cdot (2\pi)^4\, \delta^{(4)}\Big(k_1 + k_2 - \sum_{j=1}^{n} p_j\Big)}\,.$$

$\Rightarrow$ for particles with spin and other quantum #'s that φ's do not have, when calculating $|M|^2$

(i) <u>average</u> over quantum #'s of <u>initial state</u> particles, and

(ii) <u>sum</u> over quantum #'s of <u>final state</u> particles.

The LSZ Reduction Formula

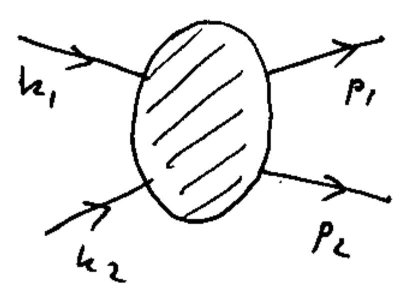
- To calculate cross sections & decay rates we need to find scattering amplitude M.

How do we find it using Feynman diagrams?

- Consider $2 \to 2$ scattering as an example:

The scattering amplitude is

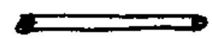
$$\underbrace{M_{2\to 2}}_{}\overset{(2\pi)^4\delta(p_1+p_2-k_1-k_2)}{=} \langle p_1, p_2 | T | k_1, k_2 \rangle.$$



What are the states $|k_1, k_2\rangle$ and $|p_1, p_2\rangle$? Obviously 2-particle states, but in the full interacting theory. It is tempting to write

$|k_1, k_2\rangle = \hat{a}^+_{\vec{k}_1} \hat{a}^+_{\vec{k}_2} |0\rangle$ with $\hat{a}^+_{\vec{k}}$ the creation operators in the interaction picture. However, while we assume no interactions at $t = \pm\infty$ <u>between</u> the particles, one can not turn off self-interactions.

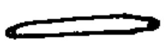
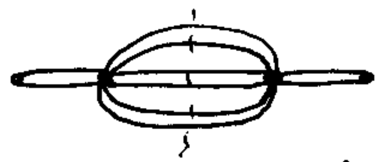
Consider 2-point function

$\int d^4x \, e^{ip\cdot x} \langle \Psi_0 | T \varphi_H(x) \varphi_H(0) | \Psi_0 \rangle \equiv$ 

In perturbation theory one has

 "dressed propagator"

It consists of 1, 2, 3, ... -particle states, where we imply physical ("dressed") particles:

 includes 

3-particle state

single-particle term

$$\int d^4x \, e^{ip\cdot x} \langle \Psi_0 | T \varphi_H(x) \varphi_H(0) | \Psi_0 \rangle = Z \frac{i}{p^2 - m_{phys}^2 + i\varepsilon}$$

+ multiparticle contributions.

$Z = 1 + O(\lambda) \sim$ normalization factor

m_{phys} is in general different from m in $\mathcal{L}$.

$\Rightarrow \varphi_H(x) \simeq \sqrt{Z} \, \varphi_{free}(x)$ at $t = \pm\infty$

$\Rightarrow \underbrace{|k_1, k_2\rangle}_{\text{dressed particles state}} = \left(\frac{1}{\sqrt{Z}}\right)^2 \cdot \hat{a}^+_{\vec{k}_1} \hat{a}^+_{\vec{k}_2} |0\rangle$ as $\varphi_H(\vec{x}, t=-\infty) \approx$

$\approx \varphi_I(\vec{x}, t=-\infty)$ in the interaction picture with $t_0 = -\infty$.

$\left(\hat{a}^{free}_{\vec{k}} = (\sqrt{Z})^{-1} \hat{a}_{\vec{k}} \right.$, with $\hat{a}_{\vec{k}}$ the usual Schrödinger creation operator$)$

Likewise $|p_1, p_2\rangle = \frac{1}{Z}\, \hat{a}_{\vec{p}_1}\, \hat{a}_{\vec{p}_2} |0\rangle$ since

$\varphi_H(\vec{x}, t=+\infty) = \varphi_I(\vec{x}, t=+\infty)$ in the interaction picture with $t_0 = +\infty$.

Consider the S-matrix:

$$\langle p_1, p_2 | S | k_1, k_2\rangle = \frac{1}{Z^2} \langle 0 | \hat{a}_{\vec{p}_1}\, \hat{a}_{\vec{p}_2}\, U(+\infty, -\infty)\, \hat{a}^+_{\vec{k}_1}\, \hat{a}^+_{\vec{k}_2} |0\rangle.$$

Write $\hat{a}^+_{\vec{k}_1} = \int d^3x\, \varphi_I(x)\, i \overleftrightarrow{\partial_0}\, e^{-ik_1\cdot x}$ ~ x^0-independent!

$(\varphi_1 \overleftrightarrow{\partial_0} \varphi_2 = \varphi_1 \partial_0 \varphi_2 - \varphi_2 \partial_0 \varphi_1)$

$\varphi_I(x)$ here is in the interaction picture with $t_0 = -\infty$.
(t_0 doesn't matter)

$$\langle p_1, p_2 | S | k_1, k_2 \rangle = \frac{1}{Z^2} \int d^3x \langle 0 | \hat{a}_{\vec{p}_1}\, \hat{a}_{\vec{p}_2}\, U(+\infty, -\infty)$$

$$\varphi_I(\vec{x}, x^0)\, \hat{a}^+_{\vec{k}_2} |0\rangle\, i \overleftrightarrow{\partial_0}\, e^{-ik_1\cdot x} = \left[\begin{array}{l}\text{as the expression is} \\ x^0\text{-independent} \Rightarrow x^0 \to -\infty\end{array}\right]$$

$$= \frac{1}{Z^2} \lim_{x^0 \to -\infty} \Big[\int d^3x \langle 0 | \hat{a}_{\vec{p}_1}\, \hat{a}_{\vec{p}_2}\, \underbrace{U(+\infty, -\infty)\, \varphi_I(\vec{x}, x^0)}_{= T\left\{\varphi_I(x)\, e^{-i\int_{-\infty}^{\infty} dt'' H_I(t'')}\right\}}\, \hat{a}^+_{\vec{k}_2} |0\rangle$$

$$\cdot\, i \overleftrightarrow{\partial_0}\, e^{-ik_1\cdot x} \Big]$$

(true at $x^0 \to -\infty$)

$$= \frac{-1}{Z^2} \int d^4x\, \partial_0 \Big[\langle 0 | \hat{a}_{\vec{p}_1}\, \hat{a}_{\vec{p}_2}\, T\left\{\varphi_I(x)\, e^{-i\int_{-\infty}^{\infty} dt'' H_I(t'')}\right\}\, \hat{a}^+_{\vec{k}_2} |0\rangle$$

$$i \overleftrightarrow{\partial_0}\, e^{-ik_1\cdot x} \Big] + \frac{1}{Z^2} \lim_{x^0 \to +\infty} \int d^3x \langle 0 | \hat{a}_{\vec{p}_1}\, \hat{a}_{\vec{p}_2}\, \underbrace{T\left\{\varphi_I(x)\, e^{-i\int_{-\infty}^{\infty} dt'' H_I(t'')}\right\}}_{\varphi_I(x)\, U(+\infty, -\infty)}$$

$$\cdot\, \hat{a}^+_{\vec{k}_2} |0\rangle\, i \overleftrightarrow{\partial_0}\, e^{-ik_1\cdot x}$$