Last time: Derived formulas for cross section:

$$d\sigma = \frac{1}{2\mathcal{E}_{k_1}\, 2\mathcal{E}_{k_2}\, |\vec{v}_1 - \vec{v}_2|} \; \underbrace{\frac{1}{n!}}_{\text{if identical particles are produced}} \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2\mathcal{E}_{p_i}} \, |M(k_1, k_2; p_1, \dots, p_n)|^2 \cdot \times (2\pi)^4 \delta^{(4)}\Big(k_1 + k_2 - \sum_{i=1}^{n} p_i\Big)$$

and for decay rate

$$d\Gamma = \frac{1}{2m} \, \frac{1}{n!} \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2\mathcal{E}_{p_i}} \, |M(k; p_1, \dots, p_n)|^2 \, (2\pi)^4 \delta^{(4)}\Big(k - \sum_{i=1}^{n} p_i\Big)$$

in terms of the scattering amplitude M.

Note: for particles with spin, etc. have to

(i) average $|M|^2$ over quantum #'s of incoming particles

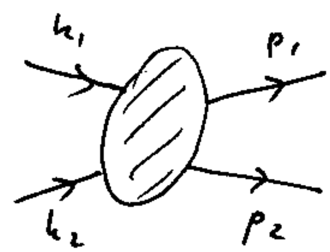
(ii) sum $|M|^2$ over — " — final state particles.

The LSZ Reduction Formula (cont'd)

How to find M? Consider $2 \to 2$ process:

$$i M_{2\to 2} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) =$$

$$= \langle p_1, p_2 | i\mathbb{T} | k_1, k_2 \rangle.$$



$$|k_1, k_2\rangle = \frac{1}{Z} \hat{a}^{+}_{\vec{k}_1} \hat{a}^{+}_{\vec{k}_2} |0\rangle \, , \quad \langle p_1, p_2| = \langle 0| \frac{1}{Z} \hat{a}_{\vec{p}_1} \hat{a}_{\vec{p}_2}$$

Z ~ a factor to remove self-dressing effects, not relevant for scattering.

Likewise $|p_1,p_2\rangle = \frac{1}{Z}\,\hat{a}_{\vec{p}_1}\,\hat{a}_{\vec{p}_2}\,|0\rangle$ since

$\varphi_H(\vec{x},t=+\infty) = \varphi_I(\vec{x},t=+\infty)$ in the interaction picture with $t_0=+\infty$.

Consider the S-matrix:

$$\langle p_1,p_2|\,S\,|k_1,k_2\rangle = \frac{1}{Z^2}\,\langle 0|\,\hat{a}_{\vec{p}_1}\,\hat{a}_{\vec{p}_2}\,U(+\infty,-\infty)\,\hat{a}^+_{\vec{k}_1}\,\hat{a}^+_{\vec{k}_2}|0\rangle.$$

Write $\hat{a}^+_{\vec{k}_1} = \int d^3x\;\varphi_I(x)\;i\overleftrightarrow{\partial_0}\;e^{-ik_1\cdot x}$ ~ x^0-independent!

$(\varphi_1\overleftrightarrow{\partial_0}\varphi_2 = \varphi_1\partial_0\varphi_2 - \varphi_2\partial_0\varphi_1)$

$\varphi_I(x)$ here is in the interaction picture with $t_0=-\infty$. (or $t_0=0$) (t_0 doesn't matter)

$$\langle p_1,p_2|\,S\,|k_1,k_2\rangle = \frac{1}{Z^2}\int d^3x\,\langle 0|\,\hat{a}_{\vec{p}_1}\,\hat{a}_{\vec{p}_2}\,U(+\infty,-\infty)$$

$$\varphi_I(\vec{x},x^0)\,\hat{a}^+_{\vec{k}_2}|0\rangle\; i\overleftrightarrow{\partial_0}\;e^{-ik_1\cdot x} = \left[\begin{array}{l}\text{as the expression is}\\ x^0\text{-independent} \Rightarrow x^0\to-\infty\end{array}\right]$$

$$= \frac{1}{Z^2}\lim_{x^0\to-\infty}\Big[\int d^3x\,\langle 0|\,\hat{a}_{\vec{p}_1}\,\hat{a}_{\vec{p}_2}\,\underbrace{U(+\infty,-\infty)\,\varphi_I(\vec{x},x^0)}_{\parallel \atop T\left\{\varphi_I(x)\,e^{-i\int_{-\infty}^{\infty}dt''H_I(t'')}\right\}}\,\hat{a}^+_{\vec{k}_2}|0\rangle$$

$$\cdot\; i\overleftrightarrow{\partial_0}\;e^{-ik_1\cdot x}\Big]$$

(true at $x^0\to-\infty$)

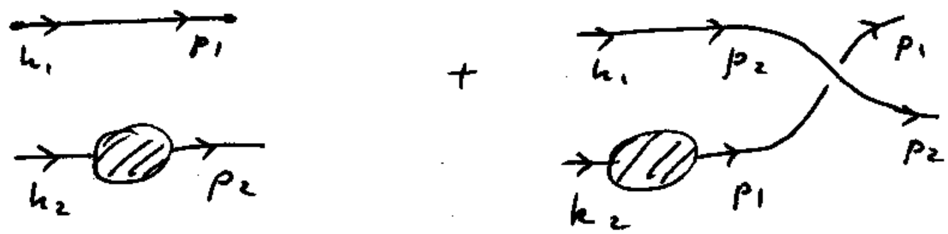
$$= \frac{-1}{Z^2}\int d^4x\;\partial_0\Big[\langle 0|\,\hat{a}_{\vec{p}_1}\,\hat{a}_{\vec{p}_2}\,T\left\{\varphi_I(x)\,e^{-i\int_{-\infty}^{\infty}dt''H_I(t'')}\right\}\hat{a}^+_{\vec{k}_2}|0\rangle$$

$$i\overleftrightarrow{\partial_0}\;e^{-ik_1\cdot x}\Big] + \frac{1}{Z^2}\lim_{x^0\to+\infty}\int d^3x\,\langle 0|\,\hat{a}_{\vec{p}_1}\,\hat{a}_{\vec{p}_2}\,\underbrace{T\left\{\varphi_I(x)\,e^{-i\int_{-\infty}^{\infty}dt''H_I(t'')}\right\}}_{\varphi_I(x)\,U(+\infty,-\infty)}$$

$$\cdot\;\hat{a}^+_{\vec{k}_2}|0\rangle\; i\overleftrightarrow{\partial_0}\;e^{-ik_1\cdot x}$$

The 2^{nd} term $= \frac{1}{Z^2} \langle 0 | \hat{a}_{\vec{p}_1} \; \hat{a}_{\vec{p}_2} \; \hat{a}^+_{\vec{k}_1} \; U(+\infty, -\infty) \, \hat{a}^+_{\vec{k}_2} | 0 \rangle$

$$= \frac{1}{Z^2} (2\pi)^3 \, 2\mathcal{E}_{k_1} \left[\delta^3(\vec{k}_1 - \vec{p}_1) \langle p_2 | S | k_2 \rangle + \delta^3(\vec{k}_1 - \vec{p}_2) \langle p_1 | S | k_2 \rangle \right]$$



$\Rightarrow$ disconnected graphs, no scattering $\Rightarrow$ drop from the cross section. We have

$$\langle p_1, p_2 | S | k_1, k_2 \rangle = \begin{matrix}\text{disconnected}\\ \text{terms}\end{matrix} + \frac{-1}{Z^2} \int d^4x \, \partial_0 \Big[\langle 0 | \hat{a}_{\vec{p}_1} \, \hat{a}_{\vec{p}_2}$$

$$\cdot T \left\{ \varphi_I(x) \, e^{-i\int_{-\infty}^{\infty} dt'' H_I(t'')} \right\} \hat{a}^+_{\vec{k}_2} | 0 \rangle \; i \overleftrightarrow{\partial_0} \, e^{-ik_1 \cdot x} \Big].$$

Since $\partial_0 \left[f(x^0) \overleftrightarrow{\partial_0} e^{-ik_1\cdot x} \right] = -\left(\partial_0^2 f(x^0)\right) e^{-ik_1 \cdot x} +$

$$+ \underbrace{f(x^0) \, \partial_0^2 \, e^{-ik_1\cdot x}}_{\substack{-\mathcal{E}_{k_1}^2 = -(\vec{k}_1^2 + m^2) = -(-\vec{\nabla}^2 + m^2) \\ = (\text{parts}) = -(-\overleftarrow{\nabla}^2 + m^2)}} = -e^{-ik_1\cdot x} \left[\partial_0^2 - \vec{\nabla}^2 + m^2 \right] f(x^0) =$$

$= -e^{-ik_1 \cdot x} (\Box + m^2) f(x^0)$. Thus

$$\langle p_1, p_2 | S | k_1, k_2 \rangle = \begin{matrix}\text{disconnected}\\ \text{terms}\end{matrix} + \frac{i}{Z^2} \int d^4x \, e^{-ik_1\cdot x} (\Box + m^2) \cdot$$

$$\cdot \langle 0 | \hat{a}_{\vec{p}_1} \, \hat{a}_{\vec{p}_2} \, T \left\{ \varphi_I(x) \, e^{-i\int_{-\infty}^{\infty} dt'' H_I(t'')} \right\} \hat{a}^+_{\vec{k}_2} | 0 \rangle$$

Repeat the procedure for other $\hat{a}^\dagger$'s & $\hat{a}$'s: at the end have

$$\boxed{\begin{aligned}\langle p_1, p_2 | S | k_1, k_2 \rangle = &\ \text{disconnected terms} + \left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \\ & e^{-ik_1\cdot x_1 - ik_2\cdot x_2 + ip_1\cdot y_1 + ip_2\cdot y_2} (\Box_{x_1} + m^2)(\Box_{x_2} + m^2)(\Box_{y_1} + m^2)\cdot \\ & \cdot(\Box_{y_2} + m^2) \langle 0 | T\left\{ \varphi_I(y_1)\varphi_I(y_2)\varphi_I(x_1)\varphi_I(x_2) e^{-i\int_{-\infty}^{\infty} dt' H_I(t')} \right\} | 0 \rangle\end{aligned}}$$

Note that $\hat{a}_{\vec{p}_2} = \int d^3y\, e^{ip_2\cdot y}\, i\overleftrightarrow{\partial_0} \varphi_I(y)$ ~ also indep. of y^0

$\Rightarrow$ for $\hat{a}$'s take $y^0 \to +\infty$ to include $\varphi_I(y)$ into time-ordering.

Finally, a small correction: our vacua are initially physical vacua, which we assume to be the same as perturbative vacuum at $t = -\infty$: $|\Psi_0(-\infty)\rangle_I = |0\rangle =$

$= |\Psi_0\rangle_H$ if $t_0 = -\infty$ in the interaction picture.

Similarly $\langle 0|$ should be replaced by interaction picture vacuum at $t = +\infty$: $\langle 0| \to {}_I\langle \Psi_0(+\infty)| = \langle 0| U(-\infty, +\infty) =$ see last quarter

$= \langle 0| U(-\infty,+\infty) \sum_n |n\rangle\langle n| = \langle 0| U(-\infty,+\infty)|0\rangle\langle 0| =$

$= \dfrac{1}{\langle 0|U(+\infty,-\infty)|0\rangle}\langle 0|$ $\Rightarrow$ have to divide the above by $\langle 0|U(+\infty,-\infty)|0\rangle$.

This would give (Gell-Mann–Low)

$$\frac{\langle 0 | T\{ \varphi_I(y_1)\varphi_I(y_2)\varphi_I(x_1)\varphi_I(x_2)\, e^{-i\int_{-\infty}^{\infty} dt' H_I(t')}\} | 0\rangle}{\langle 0 | T e^{-i\int_{-\infty}^{\infty} dt' H_I(t')} | 0\rangle} =$$

$$= \langle \Psi_0 | T\{\varphi_H(y_1)\,\varphi_H(y_2)\,\varphi_H(x_1)\,\varphi_H(x_2)\} | \Psi_0\rangle .$$

In the end we get Lehmann, Symanzik & Zimmermann (LSZ) reduction formula (1955):

$$\boxed{\begin{aligned}\langle p_1, p_2 | S | k_1, k_2\rangle = &\ \text{disconnected terms} + \left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2 \\ & e^{-ik_1\cdot x_1 - ik_2\cdot x_2 + ip_1\cdot y_1 + ip_2\cdot y_2} (\Box_{y_1}+m^2)(\Box_{y_2}+m^2)(\Box_{x_1}+m^2)(\Box_{x_2}+m^2) \\ & \cdot \langle \Psi_0 | T\{\varphi_H(y_1)\varphi_H(y_2)\varphi_H(x_1)\varphi_H(x_2)\} | \Psi_0\rangle\end{aligned}}$$

- true for any # of external legs.
- with minor modification is also true for fields with spin.
- only connected part contributes to M:

$$\boxed{\begin{aligned} iM_{2\to 2}\,(2\pi)^4\,\delta^{(4)}(k_1+k_2-p_1-p_2) = &\left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2 \\ & e^{-ik_1\cdot x_1 - ik_2\cdot x_2 + ip_1\cdot y_1 + ip_2\cdot y_2} (\Box_{y_1}+m^2)(\Box_{y_2}+m^2)(\Box_{x_1}+m^2)(\Box_{x_2}+m^2) \\ & \cdot \langle \Psi_0 | T\{\varphi_H(y_1)\varphi_H(y_2)\varphi_H(x_1)\varphi_H(x_2)\} | \Psi_0\rangle\end{aligned}}$$

~ Now we see why we wanted to calculate a time-ordered product!

$\Rightarrow$ In momentum space each factor of $i(\Box + m^2)$ becomes $\quad i(-k^2+m^2) = \dfrac{k^2-m^2}{i} = \left[\dfrac{i}{k^2-m^2}\right]^{-1}$ ~ the inverse propagator!

$\Rightarrow$ these factors remove propagators from external lines $\Rightarrow$ "truncate" the amplitude.

Also note that outgoing & incoming particles are on mass shell: $k_1^2 = k_2^2 = p_1^2 = p_2^2 = m^2$.

$\Rightarrow$ To calculate an amplitude, calculate Feynman diagram without propagators on external legs and putting all external lines on mass shell! (truncated diagram)

Feynman Rules for Scattering Amplitudes in φ^4 theory.

1. Each internal line gives $\quad \xrightarrow{k} \; = \dfrac{i}{k^2-m^2+i\varepsilon}$

2. Each vertex gives $\quad \times = -i\lambda$

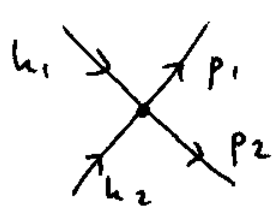
3. Each external line gives 1.

4. Impose 4-momentum conservation at each vertex. Integrate over each independent (internal) momentum $\frac{d^4k}{(2\pi)^4}$.

5. Divide by symmetry factors.

6. Keep connected diagrams only.

Example | Consider $2 \to 2$ scattering in φ^4 theory at order λ. The only contributing diagram is



$\Rightarrow$ by the rules $i\mathcal{M}_{2\to 2} = -i\lambda$.

The cross section is

identical final state particles

$$d\sigma = \frac{1}{2\mathcal{E}_{k_1}\, 2\mathcal{E}_{k_2} |\vec{v}_1 - \vec{v}_2|} \frac{d^3p_1}{(2\pi)^3 2\mathcal{E}_{p_1}} \frac{d^3p_2}{(2\pi)^3 2\mathcal{E}_{p_2}} \frac{1}{2!} |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(p_1+p_2-k_1-k_2).$$

Consider center-of-mass frame

$(\vec{k}_1 = -\vec{k}_2)$: $|\vec{v}_1 - \vec{v}_2| = \left|\frac{\vec{k}_1}{\mathcal{E}_{k_1}} - \frac{\vec{k}_2}{\mathcal{E}_{k_2}}\right| = 2\frac{|\vec{k}_1|}{\mathcal{E}_{k_1}}$ as $\mathcal{E}_{k_1} = \mathcal{E}_{k_2} = \mathcal{E}_k$.

$$d\sigma = \frac{1}{8\mathcal{E}_k |\vec{k}|} \frac{d^3p_1}{(2\pi)^3 2\mathcal{E}_{p_1}} \frac{d^3p_2}{(2\pi)^3 2\mathcal{E}_{p_2}} \frac{1}{2!} |\mathcal{M}|^2 (2\pi)^4 \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - 2\mathcal{E}_k) \cdot$$

$$\cdot \delta^{(3)}(\vec{p}_1 + \vec{p}_2) = \frac{1}{8\mathcal{E}_k |\vec{k}|} \frac{|\mathcal{M}|^2}{2!} \frac{1}{(2\pi)^2 \cdot 4\mathcal{E}_p^2} \frac{d^3p_1\, \delta(\mathcal{E}_p - \mathcal{E}_k)}{2}$$

where $\mathcal{E}_{p_1} = \mathcal{E}_{p_2} = \mathcal{E}_p$ as $\underset{\vec{p}}{\underset{\shortparallel}{\vec{p}_1}} = -\vec{p}_2$ due to δ-function.

$$d^3p\, \delta(\varepsilon_p - \varepsilon_k) = d\Omega \cdot dp \cdot p^2 \cdot \delta\left(\sqrt{p^2+m^2} - \varepsilon_k\right) =$$

$$= d\Omega\; k^2 \cdot \frac{\varepsilon_k}{k} = |\vec{k}|\varepsilon_k\, d\Omega \Rightarrow$$

$$d\sigma = \frac{1}{2!}\frac{|M|^2}{8\cancel{\varepsilon_k}\cancel{|\vec{k}|}}\;\frac{1}{8\varepsilon_k^2(2\pi)^2}\cancel{|\vec{k}|}\cancel{\varepsilon_k}\, d\Omega = \frac{1}{2!}\frac{|M|^2}{64\cdot 4\cdot \pi^2\cdot \varepsilon_k^2}\, d\Omega$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{|M|^2}{256\,\pi^2\,\varepsilon_k^2}\cdot\frac{1}{2!}}$$

It is useful to define Mandelstam variable

$s \equiv (k_1+k_2)^2$ ~ CMS energy squared

$$\Rightarrow\; s = (k_1+k_2)^2 = 4\,\varepsilon_k^2 \Rightarrow \sqrt{s} = 2\,\varepsilon_k \Rightarrow$$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{1}{2}\cdot\frac{|M|^2}{64\pi^2 s}}$$ $\Rightarrow$ for φ^4 theory we had $|M|^2 = \lambda^2 \Rightarrow$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{\lambda^2}{64\pi^2 s}\times\frac{1}{2}$$

Finally, a prediction which can be verified experimentally!

go to page 139.

Quantum Electrodynamics (QED): Tree-Level processes.

$$\mathcal{L}_{QED} = \bar{\Psi}\left[i\gamma^\mu D_\mu - m\right]\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad D_\mu = \partial_\mu + ieA_\mu$$

$\Rightarrow$ first need to find Feynman Rules for fermions & vector fields.