<u>Last time:</u> derived the LSZ reduction formula for scalars:

$$\langle p_1, p_2 | S | k_1, k_2 \rangle = \begin{matrix}\text{disconnected}\\ \text{terms}\end{matrix} + \left(\frac{i}{\sqrt{Z}}\right)^4 \int d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2$$

$$\cdot\, e^{-ik_1\cdot x_1 - ik_2\cdot x_2 + ip_1\cdot y_1 + ip_2\cdot y_2} \; (\Box_{y_1}+m^2)(\Box_{y_2}+m^2)(\Box_{x_1}+m^2)(\Box_{x_2}+m^2)$$

$$\cdot\, \langle \Psi_0 | T\{\varphi_H(y_1)\varphi_H(y_2)\,\varphi_H(x_1)\varphi_H(x_2)\} | \Psi_0 \rangle$$

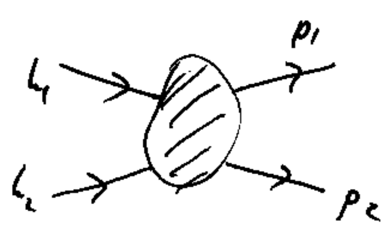
<u>Feynman Rules for scattering amplitudes in φ^4-Theory</u>

(cont'd)

1. $\xrightarrow{\;k\;} = \dfrac{i}{k^2 - m^2 + i\varepsilon}$ (internal line)
2. $\times = -i\lambda$ (vertex)
3. 1 (external line)
4. 4-momentum conservation, $\frac{d^4k}{(2\pi)^4}$ over indep. momenta
5. Symmetry factors.
6. Connected graphs only.

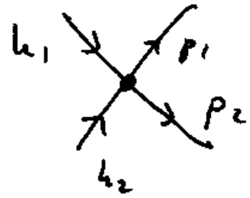
For $2\to 2$ process in CMS frame we have derived the expression for differential cross section:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{1}{2!}\,\frac{|M|^2}{256\,\pi^2 E_k^2}$$



← symmetry factor for identical particles p_1 & p_2 only!

Example



$\Rightarrow \; i\mathcal{M}_{2\to 2} = -i\lambda$

according to Feynman rules

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{1}{2!}\,\frac{\lambda^2}{256\pi^2 E_{cm}^2}$$

$$d^3p\,\delta(\mathcal{E}_p-\mathcal{E}_k) = d\Omega\cdot dp\cdot p^2\cdot\delta\left(\sqrt{p^2+m^2}-\mathcal{E}_k\right) =$$

$$= d\Omega\; k^2\cdot\frac{\mathcal{E}_k}{k} = |\vec{k}|\,\mathcal{E}_k\,d\Omega \Rightarrow$$

$$d\sigma = \frac{1}{2!}\,\frac{|M|^2}{8\cancel{\mathcal{E}_k}\cancel{|\vec{k}|}}\;\frac{1}{8\,\mathcal{E}_k^2\,(2\pi)^2}\cancel{|\vec{k}|}\cancel{\mathcal{E}_k}\,d\Omega = \frac{1}{2!}\,\frac{|M|^2}{64\cdot4\cdot\pi^2\cdot\mathcal{E}_k^2}\,d\Omega$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{|M|^2}{256\,\pi^2\,\mathcal{E}_k^2}\cdot\frac{1}{2!}}$$

It is useful to define Mandelstam variable

$s \equiv (k_1+k_2)^2$ ~ CMS energy squared

$$\Rightarrow\; s = (k_1+k_2)^2 = 4\,\mathcal{E}_k^2 \Rightarrow \sqrt{s} = 2\,\mathcal{E}_k \Rightarrow$$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{1}{2}\cdot\frac{|M|^2}{64\pi^2 s}}$$ $\Rightarrow$ for φ^4 theory we had $|M|^2 = \lambda^2 \Rightarrow$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{\lambda^2}{64\pi^2 s}\times\frac{1}{2}$$

Finally, a prediction which can be verified experimentally!

go to page 139.

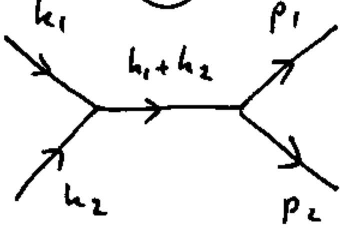
Quantum Electrodynamics (QED): Tree-Level processes.

$$\mathcal{L}_{QED} = \bar{\Psi}\left[i\gamma^\mu D_\mu - m\right]\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad D_\mu = \partial_\mu + ieA_\mu$$

$\Rightarrow$ first need to find Feynman Rules for fermions & vector fields.

Example $2 \to 2$ scattering in φ^3 theory at order λ^2 (in the amplitude). There are 3-diagrams:

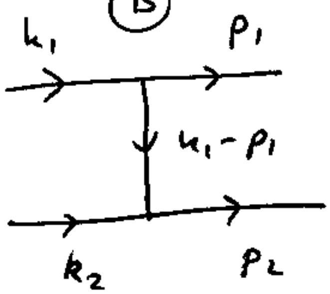
(A)



$$i M_A = (-i\lambda)^2 \frac{i}{(k_1+k_2)^2 - m^2 + i\varepsilon}$$

s-channel graph.

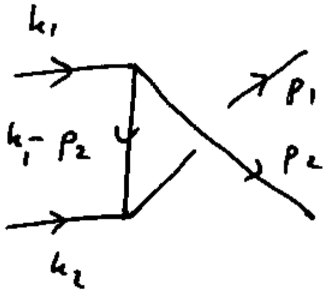
(B)



$$i M_B = (-i\lambda)^2 \frac{i}{(k_1-p_1)^2 - m^2 + i\varepsilon}$$

t-channel graph

(C)



$$i M_C = (-i\lambda)^2 \frac{i}{(k_1-p_2)^2 - m^2 + i\varepsilon}$$

u-channel graph

center-of-mass energy — momentum transfers

(Def.) <u>Mandelstam variables</u>: $\boxed{s \equiv (k_1+k_2)^2, \quad t \equiv (k_1-p_1)^2, \quad u \equiv (k_1-p_2)^2}$

Lorentz-invariant quantities,

very useful in constructing cross sections.

We write $\boxed{M_A = \dfrac{-\lambda^2}{s-m^2}}$ $\Rightarrow$ hence s-channel

$$\boxed{M_B = \frac{-\lambda^2}{t-m^2}} \Rightarrow \text{ t-channel}$$

$$\boxed{M_C = \frac{-\lambda^2}{u-m^2}} \Rightarrow \text{ u-channel.}$$

The total scattering amplitude: $M = M_A + M_B + M_C$.

The cross section:

identical final state particles.

$$d\sigma = \frac{1}{\underbrace{2\varepsilon_{k_1}\, 2\varepsilon_{k_2}\, |\vec{v}_1 - \vec{v}_2|}_{4\sqrt{(k_1\cdot k_2)^2 - m^4}}} \frac{d^3p_1}{(2\pi)^3\, 2\varepsilon_{p_1}} \frac{d^3p_2}{(2\pi)^3\, 2\varepsilon_{p_2}} \frac{1}{2!} |M|^2 (2\pi)^4 \delta^{(4)}(k_1+k_2-p_1-p_2)$$

$$\Rightarrow \boxed{d\sigma = \frac{1}{4\sqrt{(k_1\cdot k_2)^2 - m^4}} \frac{1}{2!} \frac{d^3p_1}{(2\pi)^3\, 2\varepsilon_{p_1}} \frac{d^3p_2}{(2\pi)^3\, 2\varepsilon_{p_2}} \lambda^4 \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2 \cdot (2\pi)^4 \delta^{(4)}(k_1+k_2-p_1-p_2)}$$

$$s = (k_1+k_2)^2 = 2m^2 + 2k_1\cdot k_2 \Rightarrow k_1\cdot k_2 = \frac{1}{2}s - m^2$$

$$\frac{d\sigma}{d^3p} = \frac{1}{4\sqrt{\frac{(s-2m^2)^2}{4} - m^4}} \frac{1}{2!} \int \frac{d^3p_1}{(2\pi)^3\, 2\varepsilon_{p_1}} \frac{d^3p_2}{(2\pi)^3\, 2\varepsilon_{p_2}} \lambda^4 \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2$$

$$\cdot (2\pi)^4 \delta^{(4)}(k_1+k_2-p_1-p_2) \left[\delta(\vec{p}_1 - \vec{p}) + \delta(\vec{p}_2 - \vec{p}) \right]$$

trigger on $\vec{p}_1$ trigger on $\vec{p}_2$

We get

$$\frac{d\sigma}{d^3p} = \frac{1}{2\sqrt{s(s-4m^2)}}\,\lambda^4\left[\frac{1}{s-m^2}+\frac{1}{t-m^2}+\frac{1}{u-m^2}\right]^2\frac{1}{(2\pi)^3 2\mathcal{E}_p}\,\cdot$$

$$\cdot\,\frac{1}{(2\pi)^3\,2\,\mathcal{E}_{p_2}}\cdot(2\pi)^4\,\delta\left(\mathcal{E}_{k_1}+\mathcal{E}_{k_2}-\mathcal{E}_p-\mathcal{E}_{p_2}\right)\quad\Rightarrow$$

$$\mathcal{E}_p\,\frac{d\sigma}{d^3p} = \frac{1}{2\sqrt{s(s-4m^2)}}\,\lambda^4\left[\frac{1}{s-m^2}+\frac{1}{t-m^2}+\frac{1}{u-m^2}\right]^2\frac{1}{(2\pi)^2\,4\mathcal{E}_{p_2}}\,\cdot$$

$$\cdot\,\delta\left(\mathcal{E}_{k_1}+\mathcal{E}_{k_2}-\mathcal{E}_p-\mathcal{E}_{p_2}\right)$$

note the definition of s, t, u here (see attached pp. 141', 141").

$$s+t+u = (p_1+p_2)^2+(k_2-p_2)^2+(k_1-p_2)^2 = 6m^2+2p_2\cdot(p_1-k_2-k_1)$$

$$= 6m^2 \underbrace{-2m^2}_{= p_2^2} + 2p_2\cdot(p_1+p_2-k_1-k_2) = 4m^2+2p_2\cdot(p_1+p_2-k_1-k_2)$$

We imposed 3-momentum conservation $\Rightarrow p_1+p_2-k_1-k_2 =$

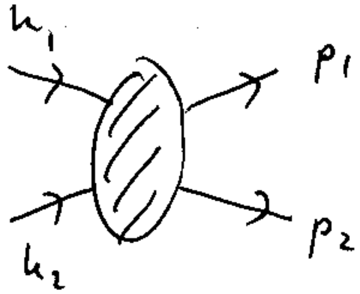
$$=\left(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{k_1}-\mathcal{E}_{k_2},\ \vec{0}\right)\ \Rightarrow\ s+t+u = 4m^2+2\mathcal{E}_{p_2}\cdot\left(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{k_1}-\mathcal{E}_{k_2}\right)$$

$$\Rightarrow\ \boxed{\frac{1}{2\mathcal{E}_{p_2}}\,\delta\left(\mathcal{E}_{k_1}+\mathcal{E}_{k_2}-\mathcal{E}_{p_1}-\mathcal{E}_{p_2}\right) = \delta\left(s+t+u-4m^2\right)}$$

$$\boxed{\mathcal{E}_p\,\frac{d\sigma}{d^3p} = \left(\frac{\lambda^2}{4\pi}\right)^2\frac{1}{\sqrt{s(s-4m^2)}}\left[\frac{1}{s-m^2}+\frac{1}{t-m^2}+\frac{1}{u-m^2}\right]^2\cdot\,\delta\left(s+t+u-4m^2\right)}$$

Lorentz-invariant form of cross section.

~ Note that Mandelstam variables are related: $s+t+u=4m^2$.
(general result)



Suppose

$$\mathcal{E}_p \frac{d\sigma}{d^3 p} = f(s,t,u)\,\frac{1}{2\mathcal{E}_{p_2}}\,\delta(\mathcal{E}_{k_1}+\mathcal{E}_{k_2}-\mathcal{E}_{p_1}-\mathcal{E}_{p_2})$$

$$s = (k_1+k_2)^2 \qquad \text{or} \qquad s' = (p_1+p_2)^2$$

$$t = (k_1-p_1)^2 \qquad\qquad t' = (k_2-p_2)^2$$

$$u = (k_1-p_2)^2 \qquad\qquad u' = (k_1-p_2)^2 = u$$

We know that $\vec{k}_1 + \vec{k}_2 = \vec{p}_1 + \vec{p}_2$.

(a) $s+t+u = 6m^2 + 2k_1\cdot(k_2-p_1-p_2) = 4m^2 + 2k_1\cdot(k_1+k_2-p_1-p_2) = \begin{pmatrix}\text{use}\\ \text{3-momentum}\\ \text{cons.}\end{pmatrix}$

$$= 4m^2 + 2\mathcal{E}_{k_1}(\mathcal{E}_{k_1}+\mathcal{E}_{k_2}-\mathcal{E}_{p_1}-\mathcal{E}_{p_2})$$

$$\Rightarrow \boxed{\frac{1}{2\mathcal{E}_{k_1}}\,\delta(\mathcal{E}_{k_1}+\mathcal{E}_{k_2}-\mathcal{E}_{p_1}-\mathcal{E}_{p_2}) = \delta(s+t+u-4m^2)} \quad (*)$$

(b) $s'+t'+u' = 6m^2 + 2p_2\cdot(p_1-k_1-k_2) = 4m^2 + 2p_2\cdot(p_1+p_2-k_1-k_2)$

$$= 4m^2 + 2\mathcal{E}_{p_2}(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{k_1}-\mathcal{E}_{k_2})$$

$$\Rightarrow \boxed{\frac{1}{2\mathcal{E}_{p_2}}\,\delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{k_1}-\mathcal{E}_{k_2}) = \delta(s'+t'+u'-4m^2)} \quad (**)$$

In general $s' \neq s$, $t' \neq t$ until energy conservation $\mathcal{E}_{p_1}+\mathcal{E}_{p_2} = \mathcal{E}_{k_1}+\mathcal{E}_{k_2}$ is imposed. After energy conservation is imposed one gets $s'=s$, $t'=t$.

This implies that $f(s,t,u)\,\delta(s+t+u-4m^2) = f(s',t',u') \cdot$
$\cdot\, \delta(s+t+u-4m^2)$

(ibid for $\delta(s'+t'+u'-4m^2)$).

In general $(*)$ and $(**)$ give

$$\boxed{\mathcal{E}_{k_1}\, \delta(s+t+u-4m^2) = \mathcal{E}_{p_2}\, \delta(s'+t'+u'-4m^2)}$$

For the cross section we then get:

$$\boxed{\begin{aligned} \mathcal{E}_p \frac{d\sigma}{d^3p} &= f(s,t,u)\; \delta(s'+t'+u'-4m^2) \\ &= f(s',t',u')\; \delta(s'+t'+u'-4m^2) \end{aligned}}$$

In other words s,t,u and s',t',u' are interchangeable outside the delta-function, but inside it they are

$$\boxed{\begin{aligned} s' &= (p_1+p_2)^2 = (\mathcal{E}_{p_1}+\mathcal{E}_{p_2})^2 - (\vec{k}_1+\vec{k}_2)^2 \\ t' &= (k_2-p_2)^2 = (\mathcal{E}_{k_2}-\mathcal{E}_{p_2})^2 - (\vec{k}_1-\vec{p})^2 \\ u' &= (k_1-p_2)^2 = (\mathcal{E}_{k_1}-\mathcal{E}_{p_2})^2 - (\vec{k}_2-\vec{p})^2 \end{aligned}}$$

where $\boxed{\mathcal{E}_{p_1} = \mathcal{E}_p = \sqrt{\vec{p}^{\,2}+m^2}}$, $\boxed{\mathcal{E}_{p_2} = \sqrt{(\vec{k}_1+\vec{k}_2-\vec{p})^2+m^2}}$.

(everything is expressed in terms of incoming momenta $\vec{k}_1$ & $\vec{k}_2$ and the tagged momentum $\vec{p}$)

As one can show, if the differential cross section is given by

$$E_p \frac{d\sigma}{d^3p} = \underbrace{f(s,t,u)}_{\text{some function}} \, \delta(s+t+u-4m^2)$$

then

$$\frac{d\sigma}{dt} = \frac{\pi}{\sqrt{s(s-4m^2)}} f(s,t,u) \quad \left(\text{with } u = 4m^2 - s - t\right)$$

(Usually s is fixed in accelerator experiments, but t & u vary collision by collision $\Rightarrow \frac{d\sigma}{dt}$ is interesting.)

We get for our cross section:

$$\frac{d\sigma}{dt} = \left(\frac{\lambda^2}{4\pi}\right)^2 \frac{\pi}{s(s-4m^2)} \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2$$

with $s+t+u = 4m^2$.

In CMS frame can use $\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{1}{128\pi^2 s} |M|^2$ derived above to get

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \left(\frac{\lambda^2}{4\pi}\right)^2 \frac{1}{8s} \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2$$

Assume s is huge, t = fixed (Regge limit) $\Rightarrow u \simeq -s$ & huge $(s \gg t) \Rightarrow$

$$\frac{d\sigma}{dt} \simeq \left(\frac{\lambda^2}{4\pi}\right)^2 \frac{\pi}{s^2} \cdot \frac{1}{t^2}$$

where we put $m=0$.

& falls off with energy s!