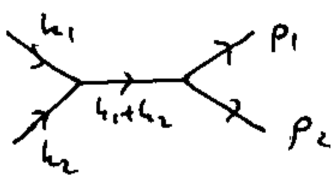
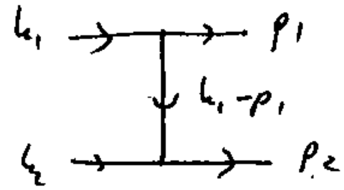
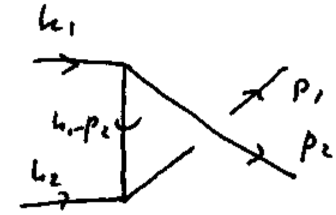
<u>Last time</u>: Finished talking about calculating cross sections in scalar theories:

<u>Example</u> $2 \to 2$ in φ^3 theory:







$s = (k_1+k_2)^2$ $t = (k_1-p_1)^2$ $u = (k_1-p_2)^2$

s-channel   t-channel   u-channel

$s, t, u \sim$ Mandelstam variables.

Showed that

$$\mathcal{E}_p \frac{d\sigma}{d^3p} = \left(\frac{\lambda^2}{4\pi}\right)^2 \frac{1}{\sqrt{s(s-4m^2)}} \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2 \cdot \frac{1}{2\mathcal{E}_{p_2}} \delta(\mathcal{E}_{k_1} + \mathcal{E}_{k_2} - \mathcal{E}_{p_1} - \mathcal{E}_{p_2})$$

$$\frac{1}{2\mathcal{E}_{p_2}} \delta(\mathcal{E}_{k_1} + \mathcal{E}_{k_2} - \mathcal{E}_{p_1} - \mathcal{E}_{p_2}) = \delta(s'+t'+u'-4m^2)$$

where $s' = (p_1+p_2)^2$, $t' = (k_2-p_2)^2$, $u' = (k_1-p_2)^2$

with $p^\mu = p_1^\mu$, $p_2^\mu = (\mathcal{E}_{k_2}, \vec{k}_1 + \vec{k}_2 - \vec{p})$.

Hence

$$\mathcal{E}_p \frac{d\sigma}{d^3p} = \left(\frac{\lambda^2}{4\pi}\right)^2 \frac{1}{\sqrt{s(s-4m^2)}} \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2 \delta(s'+t'+u'-4m^2)$$

This implies that

$$\boxed{\frac{d\sigma}{dt} = \left(\frac{\lambda^2}{4\pi}\right)^2 \frac{\pi}{s(s-4m^2)} \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2}$$

In general for $2 \to 2$ process one has

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{2!}\left(\frac{1}{4\pi}\right)^2 \frac{2\pi}{s(s-4m^2)} \langle |M|^2 \rangle}$$

symmetry factor

identical particles with mass m.

Regge limit: $s \to \infty$ ("large"), t = fixed $\Rightarrow$ $\boxed{\frac{d\sigma}{dt} \approx \left(\frac{\lambda^2}{4\pi}\right)^2 \frac{\pi}{s^2 t^2}}$

$\Rightarrow$ decreases with increasing s ~ bad for accelerators, will not see any events at high energies if all particles were spin-$\emptyset$. Luckily they are not!

Quantum Electrodynamics (QED): Tree-Level processes.

$$\mathcal{L}_{QED} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad , \quad D_\mu = \partial_\mu + ieA_\mu$$

$\Rightarrow$ first need to find Feynman Rules for fermions & vector fields.

Feynman Rules for Fermions

Free Dirac field can be decomposed as:

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k} \sum_{r=1}^{2} \left\{ \hat{b}_{\vec{k},r}\, u_r(\vec{k})\, e^{-ik\cdot x} + \hat{d}^{+}_{\vec{k},r}\, v_r(\vec{k})\, e^{ik\cdot x} \right\}.$$

Time-ordering is defined by

$$T\, \psi_\alpha(x)\, \bar{\psi}_\beta(y) = \theta(x^0 - y^0)\, \psi_\alpha(x)\, \bar{\psi}_\beta(y) - \theta(y^0 - x^0)\, \bar{\psi}_\beta(y)\, \psi_\alpha(x).$$

Feynman propagator is:

$$S_F(x-y) = \langle 0 | T\, \psi(x)\, \bar{\psi}(y) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{i(\gamma\cdot k + m)}{k^2 - m^2 + i\varepsilon}$$

Note that all operators obey anti-commutation relations!

$$\psi(x) = \underbrace{\psi^{(+)}(x)}_{\sim \hat{b}\, e^{-ik\cdot x}} + \underbrace{\psi^{(-)}(x)}_{\sim \hat{d}^{+} e^{ik\cdot x}}$$ (decompose into positive & negative frequency modes)

$$\bar{\psi} = \psi^{+}\gamma^0 = \underbrace{\bar{\psi}^{(+)}(x)}_{\sim \hat{d}\, e^{-ik\cdot x}} + \underbrace{\bar{\psi}^{(-)}(x)}_{\sim \hat{b}^{+} e^{ik\cdot x}}$$

Def. Normal ordering: move all $\hat{b}^{+}, \hat{d}^{+}$ to the left of all $\hat{b}, \hat{d}$, getting a (-1) for each interchange.

<u>Example</u>: $:\psi_\alpha(x)\,\bar{\psi}_\beta(y): \; = \; :\left(\psi_\alpha^{(+)} + \psi_\alpha^{(-)}\right)\left(\bar{\psi}_\beta^{(+)} + \bar{\psi}_\beta^{(-)}\right): \; =$

$$= \psi_\alpha^{(+)} \bar{\psi}_\beta^{(+)} + \psi_\alpha^{(-)} \bar{\psi}_\beta^{(-)} + \psi_\alpha^{(-)} \bar{\psi}_\beta^{(+)} - \bar{\psi}_\beta^{(-)} \psi_\alpha^{(+)}$$

sign change for swapping.

<u>Example:</u> $: \hat{d}_{\vec{k},r} \, \hat{b}^+_{\vec{k}',r'} : = - \hat{b}^+_{\vec{k}',r'} \hat{d}_{\vec{k},r}$

$$: \hat{d}_{\vec{k},\vec{r}} \, \hat{d}_{\vec{k}',r'} \, \hat{b}^+_{\vec{k}'',r''} : = \hat{b}^+_{\vec{k}'',r''} \hat{d}_{\vec{k},r} \hat{d}_{\vec{k}',r'} =$$

$$= -\hat{b}^+_{\vec{k}'',r''} \hat{d}_{\vec{k}',r'} \hat{d}_{\vec{k},r}$$

as $\{ \hat{d}_{\vec{k},r}, \hat{d}_{\vec{k}',r'} \} = 0$.

Def. Contraction:

$$\boxed{\wick{\c{\psi}_\alpha(x) \c{\bar{\psi}}_\beta(y)} = T\, \psi_\alpha(x) \bar{\psi}_\beta(y) - :\psi_\alpha(x) \bar{\psi}_\beta(y):}$$

Similar to the scalar field one can show that

$$\boxed{\wick{\c{\psi}_\alpha(x) \c{\bar{\psi}}_\beta(y)} = \langle 0 | T\, \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle = S_F(x-y)_{\alpha\beta}}$$

One can also show that $\boxed{\wick{\c{\psi}_\alpha(x) \c{\psi}_\beta(y)} = \wick{\c{\bar{\psi}}_\alpha(x) \c{\bar{\psi}}_\beta(y)} = 0}$.

Wick's theorem also applies, with some modifications:

$$T\, \psi_{\alpha_1}(x_1) \psi_{\alpha_2}(x_2) \bar{\psi}_{\beta_3}(x_3) \bar{\psi}_{\beta_4}(x_4) = :1\,2\,3\,4: + :\wick{\c1 1\,2\,\c1 3\,4}: +$$

$$+ :\wick{\c1 1\,2\,3\,\c1 4}: + :\wick{1\,\c1 2\,\c1 3\,4}: + :\wick{1\,\c1 2\,3\,\c1 4}: + :\wick{\c1 1\,\c2 2\,\c1 3\,\c2 4}: + :\wick{\c1 1\,\c2 2\,\c2 3\,\c1 4}: =$$

$$= :1234: - \overbracket{13}\, :24: + \overbracket{14}\, :23: + \overbracket{23}\, :14: - \overbracket{24}\, :13: -$$

$$- \overbracket{13}\;\overbracket{24} + \overbracket{14}\;\overbracket{23}$$

$\Rightarrow$ ψ only contracts with $\bar\psi$.

$\Rightarrow$ get a "$-$" from each interchange.

$\Rightarrow$ Practical consequence:

$$\boxed{\begin{aligned}&\langle 0|T\,\psi_{\alpha_1}(x_1)\,\psi_{\alpha_2}(x_2)\,\bar\psi_{\beta_3}(x_3)\,\bar\psi_{\beta_4}(x_4)|0\rangle = \\ &= -\overbracket{\psi_{\alpha_1}(x_1)\,\bar\psi_{\beta_3}(x_3)}\;\overbracket{\psi_{\alpha_2}(x_2)\,\bar\psi_{\beta_4}(x_4)} + \overbracket{\psi_{\alpha_1}(x_1)\,\bar\psi_{\beta_4}(x_4)}\cdot \\ &\quad\cdot\overbracket{\psi_{\alpha_2}(x_2)\,\bar\psi_{\beta_3}(x_3)}.\end{aligned}}$$

$\Rightarrow$ have to watch the signs.

$\Rightarrow$ applies for $\forall$ number of fields in the product.

Now imagine a theory in which fermions interact with each other (either by exchanging gauge bosons or scalar particles). LSZ reduction formula can be derived for fermions as well. Note that now:

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3\, 2E_k} \sum_{r=1}^{2} \left\{ \hat b_{\vec k, r}\, u_r(\vec k)\, e^{-ik\cdot x} + \hat d^{\,+}_{\vec k, r}\, v_r(\vec k)\, e^{ik\cdot x} \right\}$$

$$\psi^+(x) = \int \frac{d^3k}{(2\pi)^3\, 2E_k} \sum_{r=1}^{2} \left\{ \hat b^{\,+}_{\vec k, r}\, u^+_r(\vec k)\, e^{ik\cdot x} + \hat d_{\vec k, r}\, v^+_r(\vec k)\, e^{-ik\cdot x} \right\}$$

Using $u_r^+(\vec{k})\, u_{r'}(\vec{k}) = 2\mathcal{E}_k\, \delta_{rr'}$

$$v_r^+(\vec{k})\, v_{r'}(\vec{k}) = 2\mathcal{E}_k\, \delta_{rr'}$$

$$u_r^+(\vec{k})\, v_{r'}(-\vec{k}) = 0, \quad v_r^+(\vec{k})\, u_{r'}(-\vec{k}) = 0 \quad \text{(can check)}$$

we write:

on mass-shell

$$\int d^3x\, e^{ik\cdot x}\, u_r^+(\vec{k})\, \psi(x) = \int d^3x\, e^{ik\cdot x}\, u_r^+(\vec{k}) \cdot \int \frac{d^3k'}{(2\pi)^3\, 2\mathcal{E}_{k'}} \cdot$$

$$\cdot \sum_{r'=1}^{2} \left\{ \hat{b}_{\vec{k}',r'}\, u_{r'}(\vec{k}')\, e^{-ik'\cdot x} + \hat{d}^+_{\vec{k}',r'}\, v_{r'}(\vec{k}')\, e^{ik'\cdot x} \right\} =$$

$$= u_r^+(\vec{k})\, \frac{1}{2\mathcal{E}_k} \sum_{r'=1}^{2} \left\{ \hat{b}_{\vec{k},r'}\, u_{r'}(\vec{k}) + \hat{d}^+_{-\vec{k},r'}\, v_{r'}(-\vec{k}) \cdot e^{2i\mathcal{E}_k t} \right\}$$

$$= \left(\text{using the above identities}\right) = \hat{b}_{\vec{k},r}\,.$$

Using similar steps we arrive at:

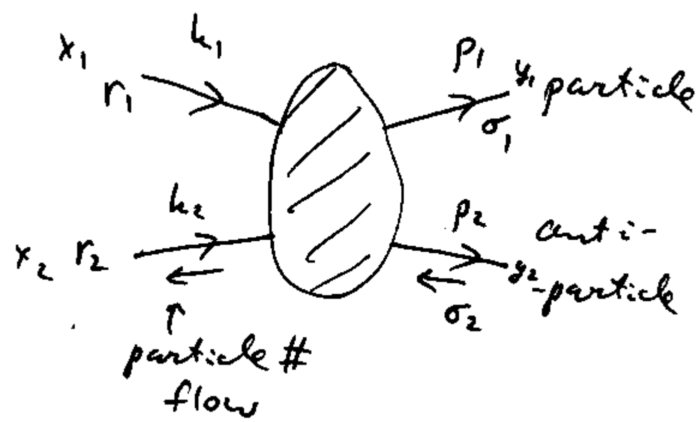
$$\boxed{\begin{aligned}
\hat{b}_{\vec{k},r} &= \int d^3x\, e^{ik\cdot x}\, \bar{u}_r(\vec{k})\, \gamma^0\, \psi(x) \\
\hat{d}^+_{\vec{k},r} &= \int d^3x\, e^{-ik\cdot x}\, \bar{v}_r(\vec{k})\, \gamma^0\, \psi(x) \\
\hat{b}^+_{\vec{k},r} &= \int d^3x\, e^{-ik\cdot x}\, \bar{\psi}(x)\, \gamma^0\, u_r(\vec{k}) \\
\hat{d}_{\vec{k},r} &= \int d^3x\, e^{ik\cdot x}\, \bar{\psi}(x)\, \gamma^0\, v_r(\vec{k})
\end{aligned}}$$

Following the steps of our LSZ derivation for scalars we get in the end the following expression for $2 \to 2$ S-matrix (bar/overline denotes anti-particles):

$$\boxed{\begin{aligned}\langle p_1, \bar{p}_2 |\, S \,| k_1, \bar{k}_2 \rangle = &\ \text{disconnected terms} + \frac{i^2(-i)^2}{(\sqrt{Z_2})^4} \cdot \int d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2\; e^{-ik_2 x_2}\, e^{ip_1 y_1} \left[\bar{v}_{r_2}(\vec{k}_2)\left(i \not{\partial}_{x_2} - m\right)\right]_{\alpha_2} \left[\bar{u}_{\sigma_1}(\vec{p}_1)\left(i \not{\partial}_{y_1} - m\right)\right]_{\beta_1} \\ &\cdot \langle \Psi_0 |\, T\left\{ \psi(x_2)_{\alpha_2}\, \bar{\psi}(y_2)_{\beta_2}\, \psi(y_1)_{\beta_1}\, \bar{\psi}(x_1)_{\alpha_1} \right\} | \Psi_0 \rangle \cdot \\ &\cdot \left[\left(-i \overleftarrow{\not{\partial}}_{x_1} - m\right) u_{r_1}(\vec{k}_1)\right]_{\alpha_1} \left[\left(-i \overleftarrow{\not{\partial}}_{y_2} - m\right) v_{\sigma_2}(\vec{p}_2)\right]_{\beta_2} e^{-ik_1 \cdot x_1 + ip_2 \cdot y_2}.\end{aligned}}$$

Here Z_2 is a new constant for fermions (different from the one for scalars).
$Z_2 \neq Z$

This is the process:



$$\not{\partial} \equiv \gamma^\mu \partial_\mu$$

(To see the result may write:

$$\hat{b}_{\vec{k},r} = \int d^4x \cdot \partial_0 \left[e^{ik\cdot x}\, \bar{u}_r(\vec{k})\, \gamma^0\, \psi(x) \right] + \text{contribution to disconnected terms} =$$

$$= \int d^4x \left\{ \bar{u}_r(\vec{k}) e^{ik\cdot x} i\gamma^0 \cdot k^0\, \psi(x) + e^{ik\cdot x}\, \bar{u}_r(\vec{k})\, \gamma^0\, \partial_0 \psi(x) \right\} + \text{disc.}$$

As $(\not{k} - m) u_r(\vec{k}) = 0 \Rightarrow \underset{\gamma^0\gamma^0 = 1}{u_r^\dagger(\vec{k})} \left[k^0\gamma^0 - \vec{k}\cdot \underset{-\vec{\gamma}}{\vec{\gamma}^{\dagger}} - m \right] = 0 \Rightarrow \bar{u}_r(\vec{k})(\not{k} - m) = 0$

$$\hat{b}_{\vec{k},r} = \int d^4x \left\{ \bar{u}_r(\vec{k})\, e^{ik\cdot x}\, i\big(\underset{=}{\vec{\gamma}\cdot\vec{k}} + m\big)\psi(x) + e^{ik\cdot x}\,\bar{u}_r(\vec{k})\cdot \right.$$

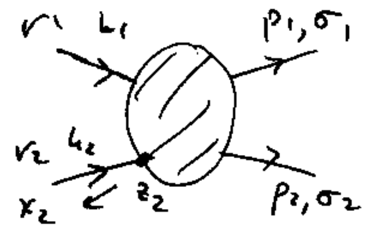
$+i\vec{\nabla}$ acting on $e^{ik\cdot x}$ $\Rightarrow$ parts

$$\left. \cdot\,\gamma^0\partial_0\,\psi(x) \right\} + \text{disc.} = \int d^4x\; e^{ik\cdot x}\,\bar{u}_r(\vec{k})\, i\left[-i\left(\gamma^0\partial_0 + \vec{\gamma}\cdot\vec{\nabla}\right) + m\right]\psi(x)$$

$$+\,\text{disc.} = \int d^4x\; e^{ik\cdot x}\,\bar{u}_r(\vec{k})(-i)\left[i\partial\!\!\!/ - m\right]\psi(x) + \text{disc.}$$

& do the same for $\hat{b}^+, \hat{d}, \hat{d}^+$.)

Again, LSZ formula can be generalized to $\forall$ number of external legs. Similar to scalars, the factors like $i[i\partial\!\!\!/ - m]$ only remove the propagators of external lines:



$$S_F(x_2 - z) = \int \frac{d^4k_2}{(2\pi)^4}\, e^{-i(-k_2)\cdot(x_2 - z)}\, \frac{i(-\gamma\cdot k_2 + m)}{k_2^2 - m^2 + i\varepsilon}$$

as k_2 flows opposite particle # flow.

$$i\left[i\partial\!\!\!/_{x_2} - m\right] e^{ik_2(x_2 - z)}\, \frac{i(-k\!\!\!/_2 + m)}{k_2^2 - m^2 + i\varepsilon} = (-i)\left[k\!\!\!/_2 + m\right] e^{ik_2(x_2 - z)} \cdot$$

$$\frac{(-i)(k\!\!\!/_2 - m)}{k_2^2 - m^2 + i\varepsilon} = -\,e^{ik_2(x_2 - z)} \quad \Rightarrow \text{"}-\text{" left.}$$

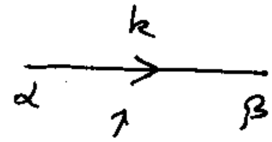
Incoming anti-particle $\Rightarrow \hat{d}^+ \Rightarrow -\bar{v}_{r_2}(\vec{k}_2)$.

Incoming particle $\Rightarrow \hat{b}^+ \Rightarrow u_{r_1}(\vec{k}_1)$

Outgoing particle $\Rightarrow \hat{b} \Rightarrow \bar{u}_{\sigma_1}(\vec{p}_1)$

— " — anti-particle $\Rightarrow \hat{d} \Rightarrow -v_{\sigma_2}(\vec{p}_2)$.

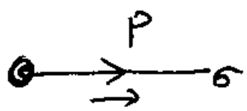
Feynman rules for fermions:

(1) For each line 

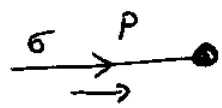
get $\dfrac{i(\not{k}+m)_{\beta\alpha}}{k^2-m^2+i\epsilon}$

particle # flow & momentum flow

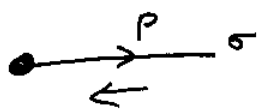
(2) External fermion lines give:



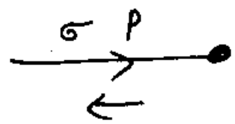
$\bar{u}_\sigma(\vec{p})$ outgoing particle



$u_\sigma(\vec{p})$ incoming particle



$v_\sigma(\vec{p})$ outgoing anti-particle



$\bar{v}_\sigma(\vec{p})$ incoming anti-particle

(3) Treat signs carefully: interchange of two identical external fermions gives a "−" sign.

Rule of thumb: associate a (-1) with:
- every closed fermion loop
- each fermion line that begins & ends in the initial (final) state.

Formulas $\sum_r u_r(\vec{k})\bar{u}_r(\vec{k}) = \not{k}+m$, $\sum_r v_r(\vec{k})\bar{v}_r(\vec{k}) = \not{k}-m$ are very useful in constructing amplitude squared.

4) Calculate symmetry factors.
(Usually $S_1 = 1$ for many theories with fermions, symmetry factor comes from S_2; can use "brute force" too.)

Feynman Rules for Gauge Bosons (photons).

Again everything is similar to scalars. Even more so that for Dirac field — as photons are bosons.

Normal ordering, contraction — all the same as for φ:

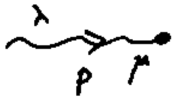
$$\overbrace{A_\mu(x)\, A_\nu(y)} = T\, A_\mu(x)\, A_\nu(y) - :A_\mu(x)\, A_\nu(y): = D_{\mu\nu}(x-y)$$

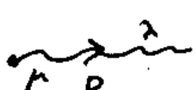
~ Feynman propagator.

LSZ reduction formula also applies. Remember that in Lorenz gauge quantization:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3\, 2\,\mathcal{E}_k} \sum_{\lambda=0}^{3} \left[\varepsilon_\mu^{(\lambda)}(\vec{k})\ \hat{a}_{\vec{k},\lambda}\, e^{-ik\cdot x} + \varepsilon_\mu^{(\lambda)*}(\vec{k}) \cdot \hat{a}^{+}_{\vec{k},\lambda} \cdot e^{ik\cdot x} \right]$$

(We included the possibility that $\varepsilon_\mu^{(\lambda)}$ is complex, e.g. for sperical polarizations.) Hence instead of u's & v's for fermions, for the photons we get:

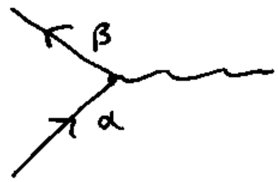
 $\varepsilon_\mu^{(\lambda)}(\vec{p})$ for incoming photon

 $\varepsilon_\mu^{(\lambda)*}(\vec{p})$ for outgoing photon.

Example $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - M)\psi + \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2 - g\varphi\bar{\psi}\psi$

Yukawa theory. ($\psi \sim$ protons, neutrons, $\varphi \sim$ pion)

Feynman rules = that for free scalars & fermions

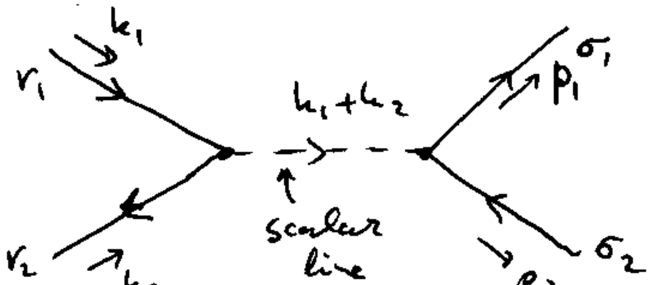
$+$ 

$= -ig\,\delta_{\alpha\beta}$ interaction vertex

Consider a process: fermion + anti-fermion $\to$

$\to$ fermion + anti-fermion.

Assume that there are 2 kinds of fermions with equal masses: protons, neutrons. Say the process is neutron + anti-neutron $\to$ proton + anti-proton. The graph is:



$$iM_{2\to 2} = (-ig)^2 \frac{i}{(k_1+k_2)^2 - m^2 + i\varepsilon} \cdot$$

$$\cdot\, \bar{u}_{\sigma_1}(p_1)\, v_{\sigma_2}(p_2)\, \bar{v}_{r_2}(k_2)\, u_{r_1}(k_1)$$

$$\Rightarrow \sum_{\sigma_1,\sigma_2,r_1,r_2} |M_{2\to 2}|^2 \cdot \underbrace{\frac{1}{4}}_{\text{average over initial helicities}} = \frac{1}{4} g^4 \frac{1}{(s-m^2)^2} \sum_{\sigma_1,\sigma_2,r_1,r_2} \bar{u}_{\sigma_1}(p_1)\, v_{\sigma_2}(p_2) \cdot$$

$$\cdot \left(\bar{u}_{\sigma_1}(p_1)\, v_{\sigma_2}(p_2)\right)^* \;\; \bar{v}_{r_2}(k_2)\, u_{r_1}(k_1) \left(\bar{v}_{r_2}(k_2)\, u_{r_1}(k_1)\right)^*$$

Start with $\sum_{\sigma_1,\sigma_2} \bar{u}_{\sigma_1}(p_1)\, v_{\sigma_2}(p_2) \big[\underbrace{\bar{u}_{\sigma_1}(p_1)}_{= u^+\gamma^0}\, v_{\sigma_2}(p_2)\big]^+$ ← can replace * with + as it is scalar

$$= \sum_{\sigma_1\sigma_2} \bar{u}_{\sigma_1}(p_1)\, v_{\sigma_2}(p_2)\, \bar{v}_{\sigma_2}(p_2)\, u_{\sigma_1}(p_1) =$$

$$= \sum_{\sigma_1} \bar{u}_{\sigma_1}(p_1)_\alpha\, (\not{p}_2 - M)_{\alpha\beta}\, u_{\sigma_1}(p_1)_\beta = (\not{p}_1 + M)_{\beta\alpha}\, (\not{p}_2 - M)_{\alpha\beta}$$

$$= \mathrm{Tr}\left[(\not{p}_1 + M)(\not{p}_2 - M)\right] = \Big|\text{as } \mathrm{Tr}\,\not{p} = \mathrm{Tr}(\gamma^\mu) p_\mu = 0 \quad = \mathrm{Tr}(\not{p}_1 \not{p}_2) - 4M^2$$

$$= p_{1\mu} p_{2\nu}\, \mathrm{Tr}(\underbrace{\gamma^\mu \gamma^\nu}_{= 4g^{\mu\nu}}) - 4M^2 = 4(p_1 \cdot p_2 - M^2).$$

Similarly $\sum_{r_1, r_2} \bar{v}_{r_2}(k_2)\, u_{r_1}(k_1) \cdot \left(\bar{v}_{r_2}(k_2)\, u_{r_1}(k_1)\right)^* = 4(k_1 \cdot k_2 - M^2).$

$$\langle |\mathcal{M}_{2\to2}|^2 \rangle = \frac{g^4}{4} \frac{1}{(s-m^2)^2} \cdot 16\, (p_1 \cdot p_2 - M^2)(k_1 \cdot k_2 - M^2).$$

Finally, as $s = (k_1 + k_2)^2 = 2M^2 + 2k_1 \cdot k_2 \Rightarrow k_1 \cdot k_2 = \frac{s}{2} - M^2.$

Similarly $p_1 \cdot p_2 = \frac{s}{2} - M^2 \Rightarrow$

$$\boxed{\langle |\mathcal{M}_{2\to2}|^2 \rangle = g^4 \frac{1}{(s-m^2)^2} \cdot (s - 4M^2)^2.}$$

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s (s - 4M^2)} \langle |\mathcal{M}_{2\to2}|^2 \rangle = \frac{g^4}{16\pi s} \frac{s - 4M^2}{(s-m^2)^2}.$$

(no $\frac{1}{2!}$ ~ different particles, no 2 ~ as t can be defined uniquely)

$$\boxed{\frac{d\sigma^{nn\to pp}}{dt} = \frac{g^4}{16\pi} \frac{s - 4M^2}{s(s-m^2)^2}}.$$