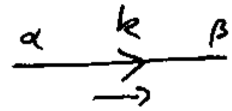
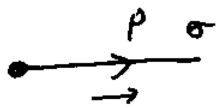
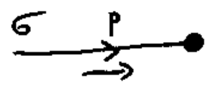
Last time: Quantum Electrodynamics (QED): Tree-Level Processes. (cont'd)

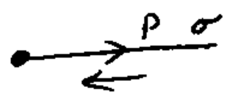
Feynman Rules for Fermions.

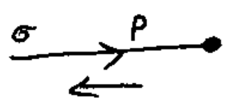
① For each internal line:  get $\dfrac{i(\not{k}+m)_{\beta\alpha}}{k^2-m^2+i\varepsilon}$.

② External fermion lines give:

 $\bar{u}_\sigma(p)$ outgoing particle

 $u_\sigma(p)$ incoming particle

 $v_\sigma(p)$ outgoing anti-particle

 $\bar{v}_\sigma(p)$ incoming anti-particle

③ (-1) for: - each fermion loop
- each fermion line that begins & ends in the initial or final state

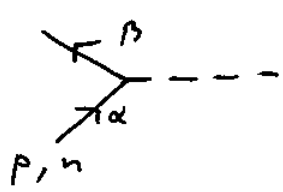
④ Symmetry factors.

Started working out an example of Yukawa th'y:

Example $\mathcal{L} = \bar{\Psi}_p(i\gamma^\mu\partial_\mu - M_p)\Psi_p + \bar{\Psi}_n(i\gamma^\mu\partial_\mu - M_n)\Psi_n$

$+\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - g\,\varphi\,\bar{\Psi}_p\Psi_p - g\,\varphi\,\bar{\Psi}_n\Psi_n$.

Interaction vertices:

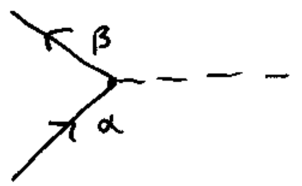
 $= -ig\,\delta_{\beta\alpha}$. (the same for protons, neutrons)

Example $\mathcal{L} = \bar{\psi}\,(i\gamma^\mu \partial_\mu - M)\psi + \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2$

$$- g\,\varphi \bar{\psi} \psi$$

Yukawa theory. (ψ ~ protons, neutrons, φ ~ pion)

Feynman rules = that for free scalars & fermions

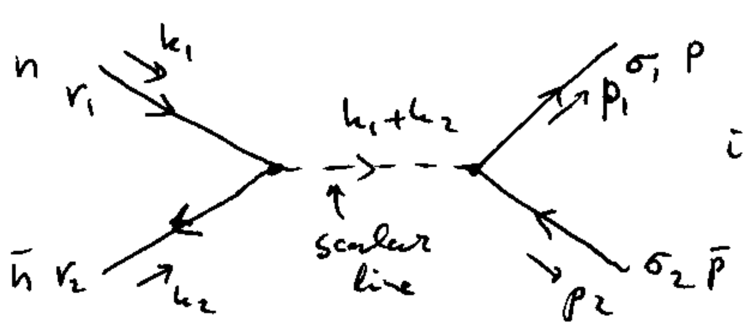
$+$ 

$= -ig\,\delta_{\alpha\beta}$ interaction vertex

Consider a process: fermion + anti-fermion →

→ fermion + anti-fermion.

Assume that there are 2 kinds of fermions with equal masses: protons, neutrons. Say the process is neutron + anti-neutron → proton + anti-proton. The graph is:



$$iM_{2\to 2} = (-ig)^2 \frac{i}{(k_1+k_2)^2 - m^2 + i\varepsilon} \cdot$$

$$\cdot\, \bar{u}_{\sigma_1}(p_1)\, v_{\sigma_2}(p_2)\, \bar{v}_{r_2}(k_2)\, u_{r_1}(k_1)$$

$$\Rightarrow \sum_{\sigma_1,\sigma_2,r_1,r_2} |M_{2\to 2}|^2 \cdot \frac{1}{4} = \frac{1}{4} g^4 \frac{1}{(s-m^2)^2} \sum_{\sigma_1,\sigma_2,r_1,r_2} \bar{u}_{\sigma_1}(p_1)\, v_{\sigma_2}(p_2) \cdot$$

average over initial helicities

$$\cdot \left(\bar{u}_{\sigma_1}(p_1)\, v_{\sigma_2}(p_2)\right)^* \; \bar{v}_{r_2}(k_2)\, u_{r_1}(k_1) \; \left(\bar{v}_{r_2}(k_2)\, u_{r_1}(k_1)\right)^*$$

Start with $\sum_{\sigma_1,\sigma_2} \bar{u}_{\sigma_1}(p_1)\, v_{\sigma_2}(p_2) \big[\underbrace{\bar{u}_{\sigma_1}(p_1)}_{= u^+\gamma^0}\, v_{\sigma_2}(p_2)\big]^+$ ← can replace * with + as it is scalar

$$= \sum_{\sigma_1,\sigma_2} \bar{u}_{\sigma_1}(p_1) v_{\sigma_2}(p_2)\, \bar{v}_{\sigma_2}(p_2)\, u_{\sigma_1}(p_1) =$$

$$= \sum_{\sigma_1} \bar{u}_{\sigma_1}(p_1)_\alpha\, (\not{p}_2 - M)_{\alpha\beta}\, u_{\sigma_1}(p_1)_\beta = (\not{p}_1 + M)_{\beta\alpha} (\not{p}_2 - M)_{\alpha\beta}$$

$$= \mathrm{Tr}\big[(\not{p}_1 + M)(\not{p}_2 - M)\big] = \Big|\text{as } \mathrm{Tr}\,\not{p} = \mathrm{Tr}(\gamma^\mu) p_\mu = 0 \quad = \mathrm{Tr}(\not{p}_1 \not{p}_2) - 4M^2$$

$$= p_{1\mu} p_{2\nu}\, \underbrace{\mathrm{Tr}(\gamma^\mu \gamma^\nu)}_{= 4g^{\mu\nu}} - 4M^2 = 4(p_1\cdot p_2 - M^2).$$

Similarly $\sum_{r_1, r_2} \bar{v}_{r_2}(k_2) u_{r_1}(k_1) \cdot \big(\bar{v}_{r_2}(k_2) u_{r_1}(k_1)\big)^* = 4(k_1\cdot k_2 - M^2)$.

$$\langle |M_{2\to 2}|^2 \rangle = \frac{g^4}{4} \frac{1}{(s-m^2)^2} \cdot 16 (p_1\cdot p_2 - M^2)(k_1\cdot k_2 - M^2).$$

Finally, as $s = (k_1 + k_2)^2 = 2M^2 + 2k_1\cdot k_2 \Rightarrow k_1\cdot k_2 = \frac{s}{2} - M^2$.

Similarly $p_1\cdot p_2 = \frac{s}{2} - M^2 \Rightarrow$

$$\langle |M_{2\to 2}|^2 \rangle = g^4 \frac{1}{(s-m^2)^2} \cdot (s - 4M^2)^2.$$

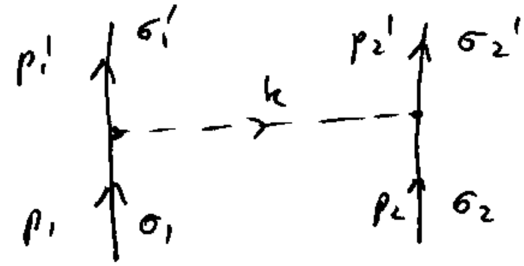
$$\frac{d\sigma}{dt} = \frac{1}{16\pi s(s-4M^2)} \langle |M_{2\to 2}|^2 \rangle = \frac{g^4}{16\pi s} \frac{s - 4M^2}{(s-m^2)^2}.$$

(no $\frac{1}{2!}$ ~ different particles, no 2 ~ as t can be defined uniquely)

$$\boxed{\frac{d\sigma}{dt}^{nn\to pp} = \frac{g^4}{16\pi} \frac{s - 4M^2}{s(s-m^2)^2}}$$

Example: Yukawa Potential

$$iM = (-ig)^2 \frac{i}{k^2 - m^2 + i\varepsilon}\, \bar{u}_{\sigma_1'}(p_1')\, u_{\sigma_1}(p_1)$$



$$\bar{u}_{\sigma_2'}(p_2')\, u_{\sigma_2}(p_2)$$

Assume protons are static & neglect recoil:

$$p_1'^{\mu} \simeq p_1^{\mu} = (M, \vec{0}) = p_2^{\mu} \simeq p_2'^{\mu}$$

$$\Rightarrow \bar{u}_{\sigma_1'}(p_1')\, u_{\sigma_1}(p_1) \simeq 2M\, \delta_{\sigma_1 \sigma_1'}, \quad \bar{u}_{\sigma_2'}(p_2')\, u_{\sigma_2}(p_2) \simeq 2M\, \delta_{\sigma_2 \sigma_2'}$$

Also, $p_1' = p_1 - k \Rightarrow (p_1')^2 = M^2 = (p_1 - k)^2 = M^2 + m_\pi^2 - 2Mk^0$

$$\Rightarrow k^0 = \frac{m_\pi^2}{2M} \simeq 0 \Rightarrow k^2 = -\vec{k}^2.$$

$$iM = \underbrace{\frac{ig^2}{\vec{k}^2 + m_\pi^2}}_{-i\tilde{V}(\vec{k})} \cdot (2m)^2\, \delta_{\sigma_1\sigma_1'}\, \delta_{\sigma_2\sigma_2'}.$$

$$\Rightarrow \tilde{V}(\vec{k}) = \frac{-g^2}{\vec{k}^2 + m_\pi^2}$$

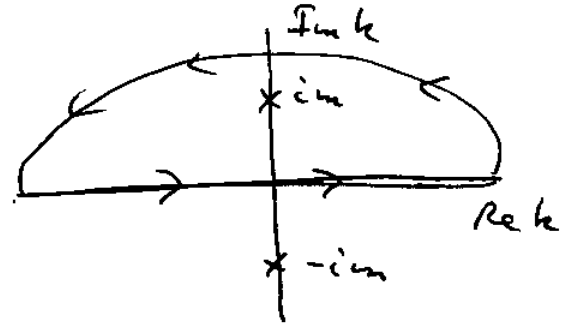
Potential between protons in momentum space.

$$V(r) = \int \frac{d^3k}{(2\pi)^3}\, e^{+i\vec{k}\cdot\vec{r}}\, \tilde{V}(\vec{k}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \frac{-g^2}{\vec{k}^2 + m_\pi^2} =$$

$$= -\frac{g^2}{(2\pi)^2} \int_0^\infty \frac{dk \cdot k^2}{k^2 + m_\pi^2} \int_{-1}^{1} d\cos\theta\, e^{ikr\cos\theta} = \frac{ig^2}{(2\pi)^2} \frac{1}{r} \cdot \int_0^\infty \frac{dk \cdot k}{k^2 + m_\pi^2} \cdot$$

$$\cdot \left(e^{ikr} - e^{-ikr}\right) =$$

$$= \frac{ig^2}{(2\pi)^2}\,\frac{1}{r}\int_{-\infty}^{\infty}\frac{dk\cdot k}{k^2+m_\pi^2}\,e^{ik\cdot r} = \text{residues} =$$



$$= \frac{ig^2}{(2\pi)^2}\,\frac{1}{r}\cdot(\cancel{2}\pi i)\,\frac{\cancel{im_\pi}}{\cancel{2im_\pi}}\,e^{-m_\pi r} =$$

$$= -\frac{g^2}{4\pi}\,\frac{e^{-m_\pi r}}{r}$$

$$\Rightarrow \boxed{V(r) = -\frac{g^2}{4\pi}\,\frac{e^{-m_\pi r}}{r}}$$

Yukawa potential

Yukawa: range of potential $\langle r\rangle \simeq \frac{1}{m_\pi}$

put $\langle r\rangle \simeq 1\,\text{fm}$ ~ internucleon distance

$\Rightarrow$ predicted a new particle (pion)

with mass $m \simeq \frac{1}{1\,\text{fm}} \simeq 200\,\text{MeV}$

(not bad, as $m_\pi \approx 140\,\text{MeV}$ in reality).

$\Rightarrow$ the potential is <u>always</u> attractive:

pp, pn, nn ...

4 Calculate symmetry factors.
(Usually $S_1 = 1$ for many theories with fermions, symmetry factor comes from S_2; can use "brute force" too.)

Feynman Rules for Gauge Bosons (photons).

Again everything is similar to scalars. Even more so that for Dirac field as photons are bosons.

Normal ordering, contraction all the same as for φ:

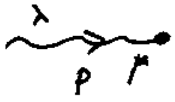
$$\overbrace{A_\mu(x)\, A_\nu}(y) = T A_\mu(x) A_\nu(y) - :A_\mu(x) A_\nu(y): = D_{\mu\nu}(x-y)$$

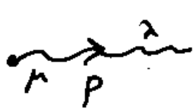
~ Feynman propagator.

LSZ reduction formula also applies. Remember that in Lorenz gauge quantization:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3\, 2E_k} \sum_{\lambda=0}^{3} \left[\varepsilon_\mu^{(\lambda)}(\vec{k})\, \hat{a}_{\vec{k},\lambda}\, e^{-ik\cdot x} + \varepsilon_\mu^{(\lambda)*}(\vec{k}) \cdot \hat{a}^\dagger_{\vec{k},\lambda} \cdot e^{ik\cdot x} \right]$$

(We included the possibility that $\varepsilon_\mu^{(\lambda)}$ is complex, e.g. for sperical polarizations.) Hence instead of u's & v's for fermions, for the photons we get:

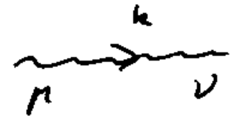
 $\varepsilon_\mu^{(\lambda)}(\vec{p})$ for incoming photon

 $\varepsilon_\mu^{(\lambda)*}(\vec{p})$ for outgoing photon.

Remember that the incoming and outgoing states must be physical $\Rightarrow$ only <u>transverse</u> polarizations contribute: $\varepsilon_\mu^{(\pm)}(\vec{k}) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$ for $\vec{k} = k\hat{z}$.

$$\begin{array}{cccc} t & x & y & z \end{array}$$

Feynman Rules for Photons

① Internal photon lines  give

$$D_{\mu\nu}(k) = \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} - (1-\lambda)\frac{k_\mu k_\nu}{k^2}\right] \text{ in Lorentz gauge.}$$

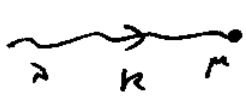
$\lambda = 0$ Landau gauge

$\lambda = 1$ Feynman gauge ~ use most of time

$$D_{\mu\nu}(k) = \frac{-i g_{\mu\nu}}{k^2+i\varepsilon}$$

② External lines:

$\varepsilon_\mu^{(\lambda)*}(\vec{k})$ outgoing photon



$\varepsilon_\mu^{(\lambda)}(\vec{k})$ incoming photon.

$\lambda = \pm$ ~ transverse only

Feynman Rules for QED.

Now we are ready to write the Feynman rules for QED. The Lagrangian is

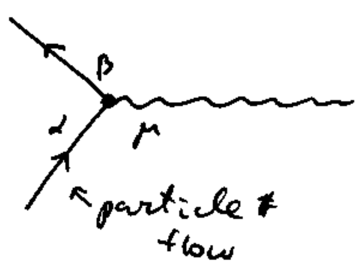
$$\mathcal{L}_{QED} = \bar{\psi}\left[i \not{D} - m\right]\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \left(\text{as } D_\mu = \partial_\mu + ieA_\mu\right)$$

$$= \underbrace{\bar{\psi}\left[i \not{\partial} - m\right]\psi}_{\text{free Dirac field}} \underbrace{- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{free photon field}} \underbrace{- e\,\bar{\psi}\gamma^\mu\psi A_\mu}_{\text{interaction}} .$$

The interaction Hamiltonian is

$$H_{int} = \int d^3x \; e\,\bar{\psi}\gamma^\mu\psi A_\mu$$

As it exponentiates $e^{-i\int dt\, H_{int}}$ $\Rightarrow$ the interaction vertex is $-ie(\gamma^\mu)_{\beta\alpha}$ and is denoted by



$\alpha, \beta \sim$ spinor indeces

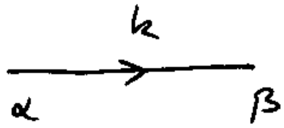
$\mu \sim$ Lorentz index

It couples photons to electrons/positrons, etc. (other fermions too).

To summarize, let us restate QED rules again:

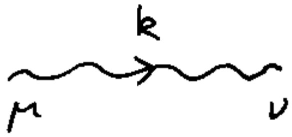
QED Feynman Rules.

(1) Each internal fermion line gives



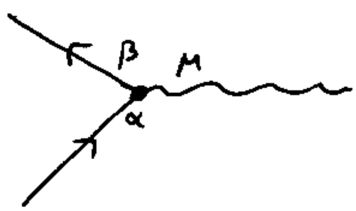
$$\frac{i\,(\not{k}+m)_{\beta\alpha}}{k^2-m^2+i\varepsilon}$$

(2) Each internal photon line gives



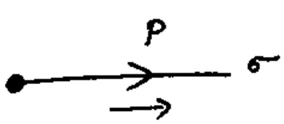
$$\frac{-i}{k^2+i\varepsilon}\,g_{\mu\nu}$$ (Feynman gauge)

(3) Photon - fermion vertex gives

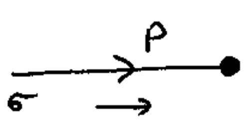


$$-ie\,(\gamma^\mu)_{\beta\alpha}$$

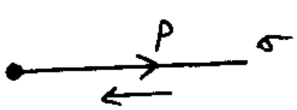
(4) External fermion lines:



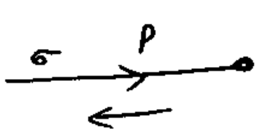
$\bar{u}_\sigma(\vec{p})$ outgoing particle



$u_\sigma(\vec{p})$ incoming particle



$v_\sigma(\vec{p})$ outgoing anti-particle



$\bar{v}_\sigma(\vec{p})$ incoming anti-particle

(5) External photon lines:

$\varepsilon_\mu^{(\lambda)*}(\vec{k})$ outgoing photon

$\varepsilon_\mu^{(\lambda)}(\vec{k})$ incoming photon

$\lambda = \pm$ ~ transverse polarizations only

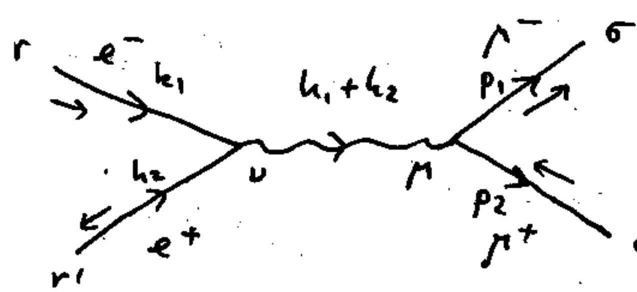
(6) (-1) for each closed fermion loop, each fermion line that begins & ends in initial ~~(final)~~ state, each diagram with fermion lines interchanged in either initial or final state.

(7) Symmetry factors.

Example: $e^+e^- \to \mu^+\mu^-$

Consider the process $e^+e^- \to \mu^+\mu^-$. It has only one Feynman diagram at the lowest order ($O(e^2)$):

The amplitude is:



$$iM = \bar{u}_\sigma(p_1)(-ie\gamma^\mu)v_{\sigma'}(p_2) \cdot \frac{-ig_{\mu\nu}}{(k_1+k_2)^2}\bar{v}_{r'}(k_2)(-ie\gamma^\nu)u_r(k_1)\,(-1) =$$

lines begin & end in initial state

$$= e^2\frac{(-i)}{s}\bar{u}_\sigma(p_1)\gamma^\mu v_{\sigma'}(p_2)\,\bar{v}_{r'}(k_2)\gamma_\mu u_r(k_1)$$