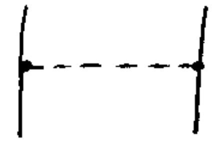
Last time | Finished discussing Yukawa theory, derived nucleon-nucleon potential:



$$\boxed{V(r) = -\frac{g^2}{4\pi}\,\frac{e^{-m_\pi r}}{r}}$$ Yukawa potential

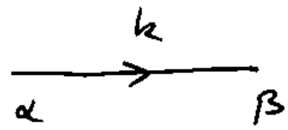
always attractive.

We derived Feynman Rules for gauge bosons (photons) and formulated Feynman Rules for QED.

To summarize, let us restate QED rules again:

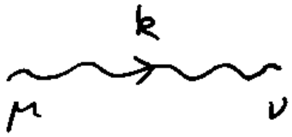
QED Feynman Rules.

① Each internal fermion line gives



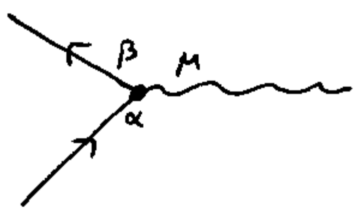
$$\frac{i\,(\not k + m)_{\beta\alpha}}{k^2 - m^2 + i\varepsilon}$$

② Each internal photon line gives



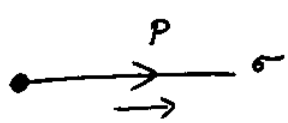
$$\frac{-i}{k^2 + i\varepsilon}\, g_{\mu\nu}$$ (Feynman gauge)

③ Photon - fermion vertex gives

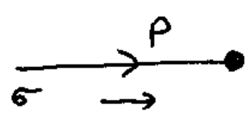


$$-ie\left(\gamma^\mu\right)_{\beta\alpha}$$

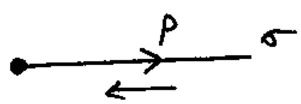
④ External fermion lines:



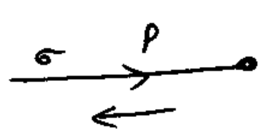
$\bar u_\sigma(\vec p)$ outgoing particle



$u_\sigma(\vec p)$ incoming particle



$v_\sigma(\vec p)$ outgoing anti-particle



$\bar v_\sigma(\vec p)$ incoming anti-particle

5. External photon lines:

$\varepsilon_\mu^{(\lambda)*}(\vec{k})$ outgoing photon

$\varepsilon_\mu^{(\lambda)}(\vec{k})$ incoming photon

$\lambda = \pm$ ~ transverse polarizations only

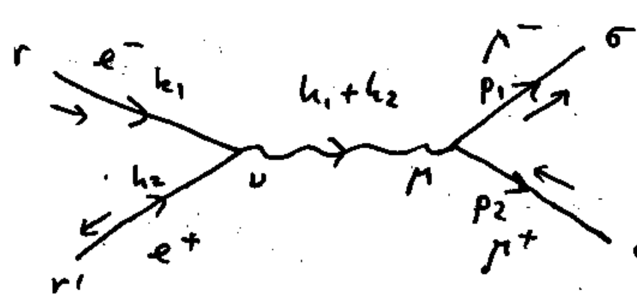
6. (-1) for each closed fermion loop, each fermion line that begins & ends in initial ~~(final)~~ state, each diagram with fermion lines interchanged in either initial/or final state.

7. Symmetry factors.

Example: $e^+e^- \to \mu^+\mu^-$

Consider the process $e^+e^- \to \mu^+\mu^-$. It has only one Feynman diagram at the lowest order ($O(e^2)$):

The amplitude is:



$$iM = \bar{u}_\sigma(p_1)(-ie\gamma^\mu)v_{\sigma'}(p_2) \cdot \frac{-ig_{\mu\nu}}{(k_1+k_2)^2}\,\bar{v}_{r'}(k_2)(-ie\gamma^\nu)u_r(k_1)\,(-1) =$$

lines begin & end in initial state

$$= e^2\frac{(-i)}{s}\,\bar{u}_\sigma(p_1)\gamma^\mu v_{\sigma'}(p_2)\,\bar{v}_{r'}(k_2)\gamma_\mu u_r(k_1)$$

Need to square the amplitude, sum over polarizations of final state particles (muons) and average over polarizations of initial state particles: need

$$\underbrace{\frac{1}{2}\sum_{r=1,2}\frac{1}{2}\sum_{r'=1,2}}_{\text{averaging}}\ \underbrace{\sum_{\sigma=1,2}\sum_{\sigma'=1,2}}_{\text{summation}} |M|^2 \equiv \langle |M| \rangle^2$$

We get: $\langle |M|^2 \rangle = \frac{1}{4}\frac{e^4}{s^2}\sum_{r,r',\sigma,\sigma'}\left[\bar{u}_\sigma(p_1)\gamma^\mu v_{\sigma'}(p_2)\,\bar{v}_{r'}(k_2)\gamma_\mu u_r(k_1)\right]$

$\cdot\left[\bar{u}_\sigma(p_1)\gamma^\nu v_{\sigma'}(p_2)\,\bar{v}_{r'}(k_2)\gamma_\nu u_r(k_1)\right]^*.$

Start with

$$\sum_{\sigma,\sigma'}\left[\bar{u}_\sigma(p_1)\gamma^\mu v_{\sigma'}(p_2)\right]\cdot\underbrace{\left[\bar{u}_\sigma(p_1)\gamma^\nu v_{\sigma'}(p_2)\right]^*}_{\text{a scalar quantity} \Rightarrow * \to +} =$$

$$= \sum_{\sigma,\sigma'}\left[\bar{u}_\sigma(p_1)\gamma^\mu v_{\sigma'}(p_2)\right]\cdot\left[u^+_\sigma(p_1)\gamma^0\gamma^\nu v_{\sigma'}(p_2)\right]^+ =$$

$$= \sum_{\sigma,\sigma'}\left[\bar{u}_\sigma(p_1)\gamma^\mu v_{\sigma'}(p_2)\right]\cdot\left[v^+_{\sigma'}(p_2)\underset{\gamma^0\gamma^0}{\wedge}(\gamma^\nu)^+\underbrace{(\gamma^0)^+}_{=\gamma^0}u_\sigma(p_1)\right]$$

$= \left(\text{as } \gamma^0(\gamma^\nu)^+\gamma^0 = \gamma^\nu\right) = \sum_{\sigma,\sigma'}\left[\bar{u}_\sigma(p_1)\gamma^\mu v_{\sigma'}(p_2)\,\bar{v}_{\sigma'}(p_2)\cdot\right.$

$\left.\cdot\gamma^\nu u_\sigma(p_1)\right] = \left(\text{sum over } \sigma' \text{ first using } \sum_{\sigma'} v_{\sigma'}(p_2)\bar{v}_{\sigma'}(p_2) = \right.$
$\left.= \not{p}_2 - m_\mu\right)$

$$= \sum_{\sigma} u_\sigma(p_1)_\alpha \, \bar{u}_\sigma(p_1)_\beta \left[\gamma^\mu(\not{p}_2 - m_\mu)\gamma^\nu\right]_{\beta\alpha} =$$

$$= (\not{p}_1 + m_\mu)_{\alpha\beta}\left[\gamma^\mu(\not{p}_2 - m_\mu)\gamma^\nu\right]_{\beta\alpha} = Tr\left[(\not{p}_1 + m_\mu)\gamma^\mu(\not{p}_2 - m_\mu)\gamma^\nu\right].$$

Similarly $\sum_{r, r'} \left[\bar{v}_{r'}(k_2)\gamma_\mu u_r(k_1)\right] \cdot \left[\bar{v}_{r'}(k_2)\gamma_\nu u_r(k_1)\right]^* =$

$$= Tr\left[(\not{k}_1 + m_e)\gamma_\nu(\not{k}_2 - m_e)\gamma_\mu\right] \text{ such that}$$

$$\langle |M|^2 \rangle = \frac{e^4}{4s^2} Tr\left[(\not{p}_1 + m_\mu)\gamma^\mu(\not{p}_2 - m_\mu)\gamma^\nu\right] \cdot$$
$$\cdot Tr\left[(\not{k}_1 + m_e)\gamma_\nu(\not{k}_2 - m_e)\gamma_\mu\right].$$

To evaluate the traces use:

$$\left\{ \begin{array}{l} Tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu} \\ Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \end{array} \right\}$$

$Tr[\gamma^\mu\gamma^\nu\gamma^\rho] = 0$ (true for any odd # of γ's)

$$Tr\left[(\not{p}_1 + m_\mu)\gamma^\mu(\not{p}_2 - m_\mu)\gamma^\nu\right] = Tr\left[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu\right] +$$

$$+ m_\mu \underbrace{\cancel{Tr[\gamma^\mu\not{p}_2\gamma^\nu]}}_{0} - m_\mu \underbrace{\cancel{Tr[\not{p}_1\gamma^\mu\gamma^\nu]}}_{0} - m_\mu^2 Tr\,\gamma^\mu\gamma^\nu =$$

$$= p_{1\alpha} p_{2\beta} Tr\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right] - m_\mu^2 \cdot 4g^{\mu\nu} =$$

$$= p_{1\alpha} p_{2\beta} \cdot 4(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta}) - m_\mu^2 \cdot 4g^{\mu\nu} =$$

$$= 4\cdot\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu} p_1\cdot p_2 - m_\mu^2 g^{\mu\nu}\right].$$

$$\mathrm{Tr}\left[(\not{k}_1 + m_e)\gamma_\nu (\not{k}_2 - m_e)\gamma_\mu\right] = (\text{similarly}) =$$

$$= 4\left[k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} - g_{\mu\nu} k_1\cdot k_2 - m_e^2 g_{\mu\nu}\right].$$

$$\langle |M|^2\rangle = \frac{4e^4}{s^2}\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}(p_1\cdot p_2 + m_\mu^2)\right]\cdot$$

$$\cdot\left[k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} - g_{\mu\nu}(k_1\cdot k_2 + m_e^2)\right] = \frac{4e^4}{s^2}\cdot$$

$$\cdot\Big[2(k_1\cdot p_1)(k_2\cdot p_2) + 2(p_1\cdot k_2)(p_2\cdot k_1) + 4\,(p_1\cdot p_2 + m_\mu^2)(k_1\cdot k_2 + m_e^2)$$

$$- 2\,p_1\cdot p_2\,(k_1\cdot k_2 + m_e^2) - 2\,k_1\cdot k_2\,(p_1\cdot p_2 + m_\mu^2)\Big] = \frac{4e^4}{s^2}\cdot$$

$$\cdot\Big[2(k_1\cdot p_1)(k_2\cdot p_2) + 2(p_1\cdot k_2)(p_2\cdot k_1) + 2m_\mu^2\, k_1\cdot k_2 + 2m_e^2\, p_1\cdot p_2$$

$$+ 4m_e^2 m_\mu^2\Big] = \frac{8e^4}{s^2}\Big[(k_1\cdot p_1)(k_2\cdot p_2) + (p_1\cdot k_2)(p_2\cdot k_1) +$$

$$+ m_\mu^2\, k_1\cdot k_2 + m_e^2\, p_1\cdot p_2 + 2m_e^2 m_\mu^2\Big] = \langle |M|^2\rangle$$

$$s = (k_1 + k_2)^2 = 2m_e^2 + 2k_1\cdot k_2 \Rightarrow k_1\cdot k_2 = \frac{s - 2m_e^2}{2};$$

$$t = (k_1 - p_1)^2 = m_e^2 + m_\mu^2 - 2k_1\cdot p_1, \qquad p_1\cdot p_2 = \frac{s - 2m_\mu^2}{2};$$

$$\Rightarrow k_1\cdot p_1 = \frac{m_e^2 + m_\mu^2 - t}{2}; \qquad k_2\cdot p_2 = \frac{m_e^2 + m_\mu^2 - t}{2}$$

$$u = (k_1 - p_2)^2 = m_e^2 + m_\mu^2 - 2k_1\cdot p_2 \Rightarrow k_1\cdot p_2 = \frac{m_e^2 + m_\mu^2 - u}{2} = k_2\cdot p_1$$

$$\langle |M|^2 \rangle = \frac{8e^4}{s^2}\left[\frac{(m_e^2+m_\mu^2-t)^2}{4} + \frac{(m_e^2+m_\mu^2-u)^2}{4} + \right.$$

$$\left. + m_\mu^2 \, \frac{s-2m_e^2}{2} + m_e^2 \, \frac{s-2m_\mu^2}{2} + 2 m_e^2 m_\mu^2 \right] = \frac{8e^4}{s^2} \cdot$$

$$\cdot \left[\frac{t^2}{4} + \frac{u^2}{4} - \frac{1}{2}\underbrace{(t+u)}_{2m_e^2+2m_\mu^2-s}(m_e^2+m_\mu^2) + \frac{m_e^4}{2} + \frac{m_\mu^4}{2} + m_e^2 m_\mu^2 \right.$$

$$\left. + m_\mu^2 \frac{s}{2} + m_e^2 \frac{s}{2} \right] = \frac{8e^4}{s^2}\left[\frac{t^2+u^2}{4} + \right.$$

$$\left. + s(m_e^2+m_\mu^2) - m_e^4 - m_\mu^4 - \cancel{2}\, m_e^2 m_\mu^2 + \frac{m_e^4}{2} + \frac{m_\mu^4}{2} + \cancel{m_e^2 m_\mu^2} \right]$$

$$= \frac{8e^4}{s^2}\left[\frac{t^2+u^2}{4} + s(m_e^2+m_\mu^2) - \frac{1}{2}(m_e^2+m_\mu^2)^2 \right] = \langle |M|^2 \rangle$$

Before we showed that

$$\frac{d\sigma}{dt} = \frac{1}{2!}\left(\frac{1}{4\pi}\right)^2 \frac{2\pi}{s(s-4m^2)} \langle |M|^2 \rangle$$

(we actually showed this for a particular case of φ^3, but true in general)

In our case no $1/2!$ $\Rightarrow$ also no 2 (distinguishable) $\Rightarrow$

$$\frac{d\sigma^{e^+e^-\to\mu^+\mu^-}}{dt} = \frac{8\pi\alpha_{EM}^2}{s(s-4m_e^2)} \, \frac{1}{s^2} \cdot \left[\frac{t^2+u^2}{4} + s(m_e^2+m_\mu^2) - \frac{1}{2}(m_e^2+m_\mu^2)^2 \right].$$

$\alpha_{EM} = \dfrac{e^2}{4\pi}$ ~ fine structure constant

We also had

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{1}{2!}\;\frac{1}{64\pi^2 s}\langle |M|^2\rangle .$$

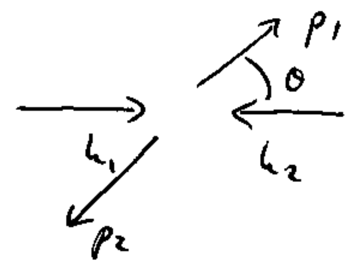
Negleting the mass of electron $(\overset{.511\,MeV}{m_e} \ll \overset{106\,MeV}{m_\mu})$ we write:

$$\left(\frac{d\sigma}{d\Omega}\right)_{cms} = \frac{1}{64\pi^2 s}\cdot\frac{2|\vec{p}|}{\underbrace{2|\vec{k}|}_{\approx\sqrt{s}\ (\text{as } m_e=0)}}\cdot\langle |M|^2\rangle \approx \frac{1}{64\pi^2 s}\;\frac{2|\vec{p}|}{\sqrt{s}}\langle |M|^2\rangle .$$

↑ correction for different particles in the final state & initial state

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{1}{64\pi^2 s}\;\frac{2|\vec{p}|}{\sqrt{s}}\cdot\frac{8e^4}{s^2}\left[\frac{t^2+u^2}{4} + s\,m_\mu^2 - \frac{1}{2}m_\mu^4\right]$$

In CMS frame:



$$t = (k_1 - p_1)^2 = -|\vec{k}_1 - \vec{p}_1|^2 =$$

$$= -\vec{k}^2 - \vec{p}^{\,2} + 2|\vec{k}|\,|\vec{p}|\cos\theta$$

$(k_1^\mu = (\mathcal{E}_k, \vec{k}),\ p_1^\mu = (\mathcal{E}_p, \vec{p}),\ \mathcal{E}_k = \mathcal{E}_p\ \ (\text{CMS frame})).$

$|\vec{p}| = \sqrt{\mathcal{E}_p^2 - m_\mu^2} = \sqrt{\mathcal{E}_k^2 - m_\mu^2} \Rightarrow$ as $s = (k_1+k_2)^2 = 4\mathcal{E}_k^2 \Rightarrow$

$\Rightarrow |\vec{p}| = \frac{1}{2}\sqrt{s - 4m_\mu^2}. \Rightarrow t = -\frac{s}{4} - \frac{1}{4}(s - 4m_\mu^2) + 2\cdot\frac{\sqrt{s}}{2}\cdot\frac{1}{2}\cdot$

$\cdot\sqrt{s-4m_\mu^2}\cos\theta .$

$$u = (k_1 - p_2)^2 = -|\vec{k}_1 - \vec{p}_2|^2 = -\vec{k}^2 - \vec{p}^{\,2} - 2|\vec{k}||\vec{p}|\cos\theta =$$

$$= -\frac{1}{2}(s - 2m_\mu^2) - \frac{1}{2}\sqrt{s(s-4m_\mu^2)}\cos\theta$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{e^4}{8\pi^2 s}\,\frac{1}{s^2}\cdot\frac{2}{2}\sqrt{1-\frac{4m_\mu^2}{s}}\cdot\left[\frac{2}{4}\left(\frac{1}{4}(s-2m_\mu^2)^2+\frac{1}{4}s(s-4m_\mu^2)\cdot\right.\right.$$

$$\left.\left.\cdot\cos^2\theta\right)+s m_\mu^2-\frac{1}{2}m_\mu^4\right] = 2\frac{\alpha_{EM}^2}{s^3}\sqrt{1-\frac{4m_\mu^2}{s}}\cdot\left[\frac{1}{8}(s-2m_\mu^2)^2+\right.$$

$$\left.+s m_\mu^2-\frac{1}{2}m_\mu^2+\frac{1}{8}s(s-4m_\mu^2)\cos^2\theta\right] = 2\frac{\alpha_{EM}^2}{s^{\cancel{3}2}}\sqrt{1-\frac{4m_\mu^2}{s}}\,\cdot$$

$$\cdot\left[\frac{1}{8}s^{\cancel{2}1}+\frac{1}{2}\cancel{s}\,m_\mu^2+\frac{1}{8}\cancel{s}(s-4m_\mu^2)\cos^2\theta\right] = \frac{\alpha_{EM}}{4\cdot s}\cdot\sqrt{1-\frac{4m_\mu^2}{s}}\,\cdot$$

$$\cdot\left[1+4\frac{m_\mu^2}{s}+\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta\right].$$

Hence

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{\alpha_{EM}^2}{4s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[1+4\frac{m_\mu^2}{s}+\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta\right]}$$

$$\sigma_{tot}^{CMS} = 2\pi\cdot\frac{\alpha_{EM}^2}{4s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[2+8\frac{m_\mu^2}{s}+\frac{2}{3}-\frac{8}{3}\frac{m_\mu^2}{s}\right]$$

$$= 2\pi\frac{\alpha_{EM}^2}{4s}\sqrt{1-\frac{4m_\mu^2}{s}}\cdot\left[\frac{8}{3}+\frac{16}{3}\frac{m_\mu^2}{s}\right] \Rightarrow$$

$$\boxed{\sigma_{tot}^{CMS} = \frac{4\pi\alpha_{EM}^2}{3s}\sqrt{1-\frac{4m_\mu^2}{s}}\cdot\left[1+2\frac{m_\mu^2}{s}\right]}.$$

$\Rightarrow$ If initial / final state particles are polarized $\Rightarrow$

$\Rightarrow$ get different θ-dependence (see e.g. Peskin) $\Rightarrow$

$\Rightarrow$ θ-dependence helps in determining polarizations in experiments.