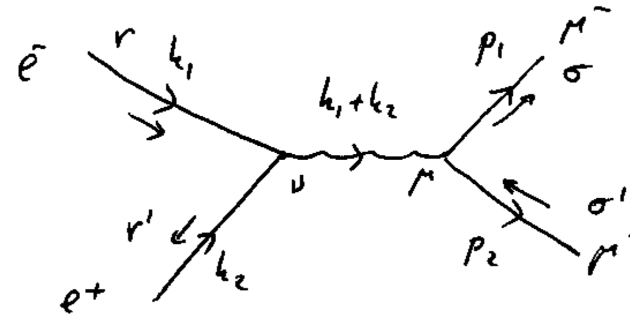
Last time | Worked out an example on calculating tree-level QED cross sections:

Example : $e^+e^- \to \mu^+\mu^-$

There is only one diagram:

(at $O(e^2)$).



We found amplitude squared:

$$\langle |M|^2 \rangle = \frac{e^4}{4s^2} \, Tr\left[(\not{p}_1 + m_\mu)\gamma^\mu (\not{p}_2 - m_\mu)\gamma^\nu \right] \cdot Tr\left[(\not{k}_2 - m_e)\gamma_\mu (\not{k}_1 + m_e)\gamma_\nu \right].$$

Evaluated the traces using: $Tr[\gamma^\mu \gamma^\nu] = 4 g^{\mu\nu}$,

$$Tr[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4\left[g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} \right],$$

$$Tr[\gamma^{\mu_1} \ldots \gamma^{\mu_{2n+1}}] = 0. \quad (\text{odd } \# \text{ of } \gamma\text{'s})$$

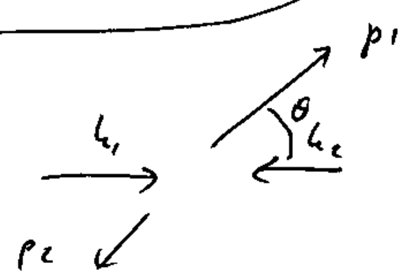
In the end obtained differential cross section

$$\frac{d\sigma^{e^+e^- \to \mu^+\mu^-}}{dt} = \frac{8\pi \, \alpha_{EM}^2}{s(s-4m_e^2)} \cdot \frac{1}{s^2} \cdot \left[\frac{t^2+u^2}{4} + s(m_e^2 + m_\mu^2) - \frac{(m_e^2+m_\mu^2)^2}{2} \right]$$

$\alpha_{EM} \equiv \frac{e^2}{4\pi}$ ~ fine structure constant.

$$\left(\frac{d\sigma}{d\Omega}\right)_{CMS} = \frac{\alpha_{EM}^2}{4s} \sqrt{1 - \frac{4m_\mu^2}{s}} \left[1 + 4\frac{m_\mu^2}{s} + \left(1 - \frac{4m_\mu^2}{s}\right)\cos^2\theta \right]$$

CMS frame X-section



different polarizations ⇒ different θ-distributions.

Feynman Rules at work: 4-pt function: (H=Heisenberg)

$$\langle \Psi_0 |\, T\{ \underset{\text{particle}}{\psi^H_{\text{muon}}(y_1)}\ \underset{\text{anti-particle}}{\bar\psi^H_{\text{muon}}(y_2)}\ \underset{\text{particle}}{\bar\psi^H_e(x_1)}\ \underset{\text{anti-particle}}{\psi^H_e(x_2)} \}\, |\Psi_0\rangle =$$

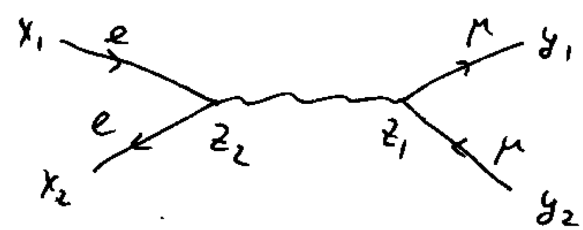
$$= \langle 0 |\, T\{ \psi^I_\mu(y_1)\, \bar\psi^I_\mu(y_2)\, \bar\psi^I_e(x_1)\, \psi^I_e(x_2)\, e^{-i\int d^4x\, e A_\rho[\bar\psi^I_\mu \gamma^\rho \psi^I_\mu + \bar\psi^I_e \gamma^\rho \psi^I_e]} \}\Big|_{\text{conn}} = -e^2 \langle 0 |\, T\{ \psi^I_\mu(y_1)\, \bar\psi^I_\mu(y_2)\, \bar\psi^I_e(x_1)\, \psi^I_e(x_2) \cdot$$

$$\cdot A^I_\nu(z_1) A_\rho(z_2)\, \bar\psi^I_\mu(z_1)\gamma^\nu \psi^I_\mu(z_1)\, \bar\psi^I_e(z_2)\gamma^\rho \psi^I_e(z_2) \}\, |0\rangle + O(e^3)$$

$$= -e^2\, \psi^I_\mu(y_1)\, \bar\psi^I_\mu(y_2)\, \bar\psi^I_e(x_1)\, \psi^I_e(x_2)\, \bar\psi^I_\mu(z_1)\gamma^\nu \psi^I_\mu(z_1)\, \bar\psi^I_e(z_2)\gamma^\rho \psi^I_e(z_2)\, A_\nu(z_1) A_\rho(z_2) = -e^2\, S^\mu_F(y_1 - z_1)\cdot \gamma^\nu$$

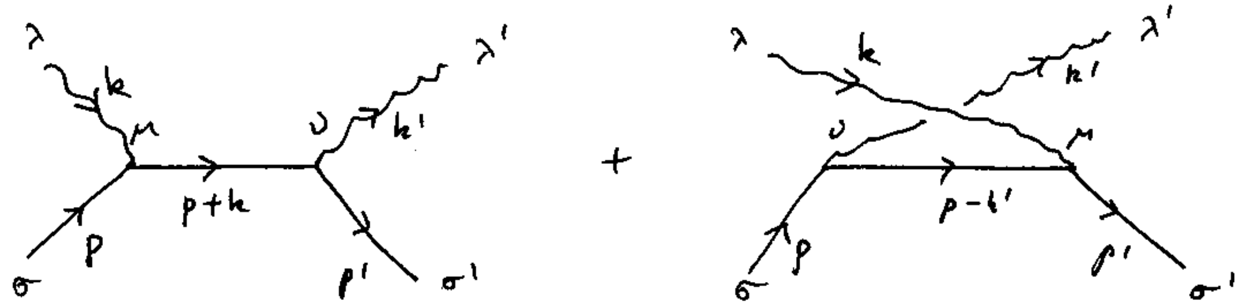
$$\cdot S^\mu_F(z_1 - y_2)\ S_F(x_2 - z_2)\,\gamma^\rho\, S_F(z_2 - x_1)\ D_{\nu\rho}(z_1 - z_2)$$

corresponding to this diagram:



LSZ make us truncate external legs propagators & replace them with u's & v's.

Compton Scattering : $e^-\gamma \to e^-\gamma$



$$iM_{e^-\gamma\to e^-\gamma} = (-ie)^2\,\varepsilon_\nu^{(\lambda')*}(k')\,\varepsilon_\mu^{(\lambda)}(k)\cdot\Big[\bar u_{\sigma'}(p')\gamma^\nu\frac{i}{\not p+\not k-m}\gamma^\mu\cdot u_\sigma(p) + \bar u_{\sigma'}(p')\gamma^\mu\frac{i}{\not p-\not k'-m}\gamma^\nu u_\sigma(p)\Big]$$

Need to square & sum/average: $\frac{1}{4}\sum_{\lambda,\sigma,\lambda',\sigma'}|M|^2$.

What to do with ε_μ & ε_ν^*? Prescription:

$$\boxed{\sum_{\lambda=\pm}\varepsilon_\mu^{(\lambda)*}(k)\,\varepsilon_\nu^{(\lambda)}(k) \to -g_{\mu\nu}}$$

Not an identity, usually also get terms $\sim k_\mu, k_\nu$, but those vanish when multiplying amplitudes due to gauge invariance.

$$\frac{1}{4}\sum_{\substack{\lambda,\lambda'\\ \sigma,\sigma'}}|M|^2 = \frac{1}{4}e^4\sum_{\sigma,\sigma'} g_{\mu\mu'}\,g_{\nu\nu'}\Big[\bar u_{\sigma'}(p')\gamma^\nu\frac{1}{\not p+\not k-m}\gamma^\mu u_\sigma(p) + \bar u_{\sigma'}(p')\gamma^\mu\frac{1}{\not p-\not k'-m}\gamma^\nu u_\sigma(p)\Big]\cdot\Big[\bar u_{\sigma'}(p')\gamma^{\nu'}\frac{1}{\not p+\not k-m}\gamma^{\mu'}u_\sigma(p) + \bar u_{\sigma'}(p')\gamma^{\mu'}\cdot\frac{1}{\not p-\not k'-m}\gamma^{\nu'}u_\sigma(p)\Big]^* = \frac{e^4}{4}\cdot\Big\{\frac{1}{((p+k)^2-m^2)^2}\overset{①}{\mathrm{Tr}}\Big[(\not p'+m)\gamma^\nu(\not p+\not k+m)\cdot$$

$$\cdot\gamma^{\mu}(\not p+m)\gamma_{\mu}(\not p+\not k+m)\gamma_{\nu}\Big] + \frac{1}{[(p-k')^2-m^2]^2}\cdot \overset{②}{\mathrm{Tr}}\Big[(\not p'+m)\gamma^{\mu}\cdot$$

$$(\not p-\not k'+m)\gamma^{\nu}(\not p+m)\gamma_{\nu}(\not p-\not k'+m)\gamma_{\mu}\Big] + \frac{1}{[(p+k)^2-m^2][(p-k')^2-m^2]}\cdot$$

$$\cdot\Big(\overset{③}{\mathrm{Tr}}\Big[(\not p'+m)\gamma^{\nu}(\not p+\not k+m)\gamma^{\mu}(\not p+m)\gamma_{\nu}(\not p-\not k'+m)\gamma_{\mu}\Big] +$$

$$+\overset{④}{\mathrm{Tr}}\Big[(\not p'+m)\gamma^{\mu}(\not p-\not k'+m)\gamma^{\nu}(\not p+m)\gamma_{\mu}(\not p+\not k+m)\gamma_{\nu}\Big)\Big\}$$

Using $\boxed{\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_n}] = \mathrm{Tr}[\gamma^{\mu_n}\gamma^{\mu_{n-1}}\ldots\gamma^{\mu_2}\gamma^{\mu_1}]}$

we see that ③ = ④. Start evaluating:

$$① = \mathrm{Tr}\Big[(\not p'+m)\gamma^{\nu}(\not p+\not k+m)\gamma^{\mu}(\not p+m)\gamma_{\mu}(\not p+\not k+m)\gamma_{\nu}\Big] \Rightarrow$$

using $\boxed{\gamma^{\mu}\gamma^{\rho}\gamma_{\mu} = -2\gamma^{\rho}}$ get (also use $\gamma_{\mu}\gamma^{\mu}=4$)

$$① = \mathrm{Tr}[\not p' \gamma^\nu (\not p + \not k)\gamma^\mu \not p \gamma_\mu (\not p + \not k)\gamma_\nu] + m^2 \{\mathrm{Tr}[\gamma^\nu \gamma^\mu \cdot$$
$$\cdot \not p \gamma_\mu (\not p + \not k)\gamma_\nu] + \mathrm{Tr}[\gamma^\nu (\not p + \not k)\gamma^\mu \gamma_\mu (\not p + \not k)\gamma_\nu] +$$
$$+ \mathrm{Tr}[\gamma^\nu (\not p + \not k)\gamma^\mu \not p \gamma_\mu \gamma_\nu] + \mathrm{Tr}[\not p' \gamma^\nu \gamma^\mu \not p \gamma_\mu \gamma_\nu] +$$
$$+ \mathrm{Tr}[\not p' \gamma^\nu \gamma^\mu \gamma_\mu (\not p + \not k)\gamma_\nu] + \mathrm{Tr}[\not p' \gamma^\nu (\not p + \not k)\gamma^\mu \gamma_\mu \gamma_\nu]\}$$
$$+ m^4 \mathrm{Tr}[\gamma^\nu \gamma^\mu \gamma_\mu \gamma_\nu] = 4\,\mathrm{Tr}[\not p' (\not p + \not k)\not p (\not p + \not k)] +$$
$$+ m^2 \{ 4\cdot(-2)\cdot 4\, \underline{p\cdot(p+k)} + 4\cdot 4\cdot 4\cdot (p+k)^2 + (-2)\cdot 4\cdot \underline{4\cdot p\cdot(p+k)}$$
$$+ (-2)^2\cdot 4\, p\cdot p' + 4\cdot(-2)\cdot 4\, \underline{\underline{p'\cdot(p+k)}} + 4\cdot(-2)\cdot 4\cdot \underline{\underline{p'\cdot(p+k)}} \}$$
$$+ 4^3 m^4 = 16\{ 2\, p\cdot(p+k)\, p'\cdot(p+k) - p\cdot p' (p+k)^2 +$$
$$+ m^2 [\underbrace{-4\, p\cdot(p+k) + 4(p+k)^2}_{4k\cdot(p+k) = 4p\cdot k \quad \text{as } k^2 = 0.} - 4p'\cdot(p+k) + p\cdot p'] + 4m^4 \}$$
$$= 16\cdot\{ \underline{2m^2 p\cdot p'} + 2m^2 p'\cdot k + \cancel{2p\cdot k\, p\cdot p'} + 2p\cdot k\, p'\cdot k -$$
$$- \underline{m^2 p\cdot p'} \cancel{-2p\cdot p'\, p\cdot k} + 4m^2 p\cdot k - \underline{3m^2 p\cdot p'} - 4m^2 p'\cdot k + 4m^4 \}$$
$$= 16\{ 2p\cdot k\, p'\cdot k + 4m^2 p\cdot k - 2m^2 p'\cdot k - 2m^2 p\cdot p' + 4m^4 \}$$

Now, $s = (p+k)^2 \Rightarrow p\cdot k = \frac{s-m^2}{2}$

$$t = (p-p')^2 \Rightarrow p\cdot p' = \frac{2m^2 - t}{2}$$

$$u = (p-k')^2 = (p'-k)^2 \Rightarrow p'\cdot k = \frac{m^2-u}{2}$$

Hence ① $= 16\left\{-\frac{1}{2}(s-m^2)(u-m^2) + 2m^2(s-m^2) + m^2(u-m^2) + m^2(t-2m^2) + 4m^4\right\} = 16\left\{-\frac{1}{2}(s-m^2)(u-m^2) + 2m^2(s-m^2) + m^2(\underbrace{t+u-3m^2}_{-s-m^2}) + 4m^4\right\}.$

$$\text{①} = 16\left\{-\frac{1}{2}(s-m^2)(u-m^2) + m^2(s-m^2) + 2m^4\right\}$$

To get ② replace $k \to -k' \Rightarrow s \leftrightarrow u \Rightarrow$

$$\text{②} = 16\left\{-\frac{1}{2}(s-m^2)(u-m^2) + m^2(u-m^2) + 2m^4\right\}.$$

Skipping the algebra we just write:

$$\text{③} = \text{④} = +8\left\{4m^4 + m^2(s-m^2) + m^2(u-m^2)\right\}.$$

$$\langle|M|^2\rangle = 4e^4\left\{\frac{1}{(s-m^2)^2}\left[-\frac{1}{2}(s-m^2)(u-m^2) + m^2(s-m^2) + 2m^4\right] + \frac{1}{(u-m^2)^2}\left[-\frac{1}{2}(s-m^2)(u-m^2) + m^2(u-m^2) + 2m^4\right] + \frac{1}{(s-m^2)(u-m^2)}\left[4m^4 + m^2(s-m^2) + m^2(u-m^2)\right]\right\} = 4e^4\left\{-\frac{1}{2}\left(\frac{u-m^2}{s-m^2} + \frac{s-m^2}{u-m^2}\right) + \cancel{\frac{m^2}{s-m^2}} + \cancel{\frac{m^2}{u-m^2}} + \cancel{\frac{m^2}{u-m^2}} + \cancel{\frac{m^2}{s-m^2}} + 2m^4\left[\frac{1}{(s-m^2)^2} + \frac{1}{(u-m^2)^2} + 2\,\frac{1}{s-m^2}\,\frac{1}{u-m^2}\right]\right\} = 4e^4\left\{-\frac{1}{2}\left(\frac{u-m^2}{s-m^2} + \frac{s-m^2}{u-m^2}\right) + 2m^4\left[\frac{1}{s-m^2} + \frac{1}{u-m^2}\right]^2 + 2m^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)\right\}$$