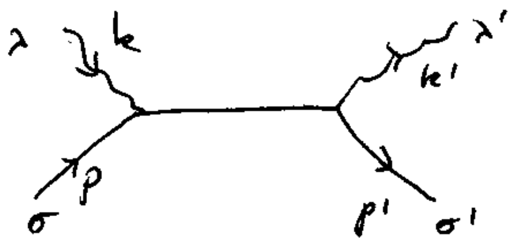
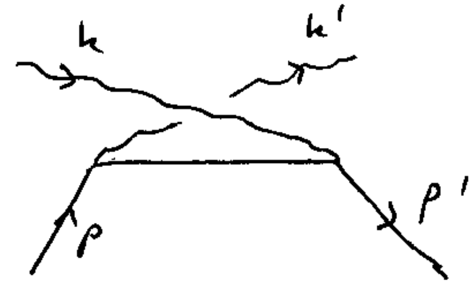
## Last time | Compton Scattering: $e^-\gamma \to e^-\gamma$




photon polarizations: $\displaystyle\sum_{\lambda=\pm} \varepsilon_\mu^{(\lambda)*}(k)\, \varepsilon_\nu^{(\lambda)}(k) \to -g_{\mu\nu}$

$$\frac{1}{4}\sum_{\substack{\lambda,\lambda'\\ \sigma,\sigma'}} |M|^2 \equiv \langle |M|^2\rangle = \frac{e^4}{4}\Big\{ \frac{1}{(s-m^2)^2} \overset{①}{\mathrm{Tr}}\big[(\not p'+m)\gamma^\nu(\not p+\not k+m)\gamma^\mu$$

$$\cdot(\not p+m)\gamma_\mu(\not p+\not k+m)\gamma_\nu\big] + \frac{1}{(u-m^2)^2}\overset{②}{\mathrm{Tr}}\big[(\not p'+m)\gamma^\mu(\not p-\not k'+m)\gamma^\nu\cdot$$

$$\cdot(\not p+m)\gamma_\nu(\not p-\not k'+m)\gamma_\mu\big] + \frac{1}{(s-m^2)(u-m^2)}\Big[\overset{③}{\mathrm{Tr}}\big[(\not p'+m)\gamma^\nu(\not p+\not k+m)\gamma^\mu$$

$$\cdot(\not p+m)\gamma_\nu(\not p-\not k'+m)\gamma_\mu\big] + \overset{④}{\mathrm{Tr}}\big[(\not p'+m)\gamma^\mu(\not p-\not k'+m)\gamma^\nu(\not p+m)\gamma_\mu\cdot$$

$$\cdot(\not p+\not k+m)\gamma_\nu\big]\Big]\Big\}.$$

Using $\gamma_\mu\gamma^\mu = 4$, $\gamma^\mu\gamma^\rho\gamma_\mu = -2\gamma^\rho$ we got

$$① = 16\left\{-\frac{1}{2}(s-m^2)(u-m^2) + m^2(s-m^2) + 2m^4\right\}$$

To find ② replace $k \to -k'$ $\Rightarrow$ $s \leftrightarrow u$ $\Rightarrow$

$$② = 16\left\{-\frac{1}{2}(s-m^2)(u-m^2) + m^2(u-m^2) + 2m^4\right\}$$

Hence ① $= 16\left\{-\frac{1}{2}(s-m^2)(u-m^2) + 2m^2(s-m^2) + m^2(u-m^2) + m^2(t-2m^2) + 4m^4\right\} = 16\left\{-\frac{1}{2}(s-m^2)(u-m^2) + \right.$

$\left. + 2m^2(s-m^2) + m^2(\underbrace{t+u-3m^2}_{-s-m^2}) + 4m^4\right\}.$

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To get ② replace $k \to -k' \Rightarrow s \leftrightarrow u \Rightarrow$

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Skipping the algebra we just write:

$$③ = ④ = +8\left\{4m^4 + m^2(s-m^2) + m^2(u-m^2)\right\}.$$

$$\langle |M|^2\rangle = 4e^4\left\{\frac{1}{(s-m^2)^2}\left[-\frac{1}{2}(s-m^2)(u-m^2) + m^2(s-m^2) + 2m^4\right]\right.$$

$$+ \frac{1}{(u-m^2)^2}\left[-\frac{1}{2}(s-m^2)(u-m^2) + m^2(u-m^2) + 2m^4\right] + \frac{1}{(s-m^2)(u-m^2)}$$

$$\left.\cdot\left[4m^4 + m^2(s-m^2) + m^2(u-m^2)\right]\right\} = 4e^4\left\{-\frac{1}{2}\left(\frac{u-m^2}{s-m^2} + \frac{s-m^2}{u-m^2}\right)\right.$$

$$+ \cancel{\frac{m^2}{s-m^2}} + \cancel{\frac{m^2}{u-m^2}} + \cancel{\frac{m^2}{u-m^2}} + \cancel{\frac{m^2}{s-m^2}} + 2m^4\left[\frac{1}{(s-m^2)^2} + \frac{1}{(u-m^2)^2}\right.$$

$$\left.\left. + 2\frac{1}{s-m^2}\frac{1}{u-m^2}\right]\right\} = 4e^4\left\{-\frac{1}{2}\left(\frac{u-m^2}{s-m^2} + \frac{s-m^2}{u-m^2}\right) + 2m^4\left[\frac{1}{s-m^2} + \right.\right.$$

$$\left.\left. + \frac{1}{u-m^2}\right]^2 + 2m^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)\right\}$$

$$\langle |M|^2\rangle = 4e^4\left\{-\frac{1}{2}\left(\frac{u-m^2}{s-m^2}+\frac{s-m^2}{u-m^2}\right)+2m^2\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right) + 2m^4\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right)^2\right\}$$

Lab frame = rest frame of initial electron =>

$$p^\mu=(m,\vec{0}) \Rightarrow s = (p+k)^2 = m^2+2m\,\mathcal{E}_k$$

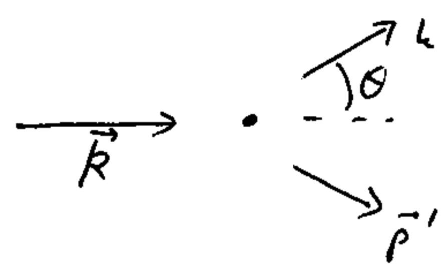
$$u = (p-k')^2 = m^2-2m\,\mathcal{E}_{k'}$$

$$\langle |M|^2\rangle = 4e^4\left\{\frac{1}{2}\left(\frac{\mathcal{E}_k}{\mathcal{E}_{k'}}+\frac{\mathcal{E}_{k'}}{\mathcal{E}_k}\right)+m\left(\frac{1}{\mathcal{E}_k}-\frac{1}{\mathcal{E}_{k'}}\right)+\frac{m^2}{2}\cdot\left(\frac{1}{\mathcal{E}_k}-\frac{1}{\mathcal{E}_{k'}}\right)^2\right\}.$$

Now, $p+k=p'+k' \Rightarrow p'=p+k-k' \Rightarrow m^2=(p')^2=(p+k-k')^2=$

$$= m^2+2m(\mathcal{E}_k-\mathcal{E}_{k'})+(k-k')^2 = m^2+2m(\mathcal{E}_k-\mathcal{E}_{k'})-2k\cdot k'$$

$$0 = \cancel{2}m(\mathcal{E}_k-\mathcal{E}_{k'})-\cancel{2}\,\mathcal{E}_k\mathcal{E}_{k'}+\cancel{2}\,\mathcal{E}_k\mathcal{E}_{k'}\cos\theta$$




$$\Rightarrow 0 = m\left(\frac{1}{\mathcal{E}_{k'}}-\frac{1}{\mathcal{E}_k}\right)-1+\cos\theta$$

$$\frac{1}{\mathcal{E}_{k'}}-\frac{1}{\mathcal{E}_k} = \frac{1}{m}(1-\cos\theta)$$

$$\Rightarrow \langle |M|^2\rangle = 2e^4\left\{\frac{\mathcal{E}_k}{\mathcal{E}_{k'}}+\frac{\mathcal{E}_{k'}}{\mathcal{E}_k}\underbrace{-2(1-\cos\theta)+(1-\cos\theta)^2}_{-1+\cos^2\theta\,=\,-\sin^2\theta}\right\}$$

$$\Rightarrow \langle |M|^2\rangle_{Lab} = 2e^4\left\{\frac{\mathcal{E}_k}{\mathcal{E}_{k'}}+\frac{\mathcal{E}_{k'}}{\mathcal{E}_k}-\sin^2\theta\right\}.$$

The cross section is:

$$d\sigma = \frac{1}{2m\,2\mathcal{E}_K\,\underbrace{|\vec{v}_k - \vec{0}|}_{=1}}\;\frac{d^3k'}{(2\pi)^3 2\mathcal{E}_{k'}}\;\frac{d^3p'}{(2\pi)^3 2\mathcal{E}_{p'}}\;\langle |M|^2\rangle_{Lab}\,(2\pi)^4\cdot$$

$$\cdot\,\delta^{(4)}(p'+k'-p-k) = \frac{1}{2m\,2\mathcal{E}_k}\;\cdot\;\frac{1}{(2\pi)^3}\cdot d\Omega\,\cdot$$

$$\cdot\int_0^\infty \frac{dk'\cdot k'^2}{2\mathcal{E}_{k'}\,2\mathcal{E}_{p'}}\langle |M|^2\rangle_{Lab}\;2\pi\,\delta(\mathcal{E}_{p'}+\mathcal{E}_{k'}-m-\mathcal{E}_K) =$$

$$\mathcal{E}_{p'} = \sqrt{m^2+(\vec{k}-\vec{k}')^2} = \sqrt{m^2+\vec{k}^2+\vec{k}'^2-2kk'\cos\theta}$$

$$= \frac{1}{4m\mathcal{E}_k}\,\frac{1}{(2\pi)^2}\,d\Omega\cdot\frac{k'^2}{4\mathcal{E}_{k'}\mathcal{E}_{p'}}\cdot\frac{1}{\left|1+\dfrac{k'-k\cos\theta}{\mathcal{E}_{p'}}\right|_{\text{conserved energy now}}}\langle |M|^2\rangle_{Lab}$$

$$= \frac{1}{4m\mathcal{E}_k}\,\frac{1}{(2\pi)^2}\,d\Omega\;\frac{k'^2}{4\mathcal{E}_{k'}}\;\frac{1}{|\underbrace{\mathcal{E}_{p'}+\mathcal{E}_{k'}}_{m+\mathcal{E}_k}-\mathcal{E}_k\cos\theta|}\;\langle |M|^2\rangle_{Lab}$$

$$= \frac{1}{4m\mathcal{E}_k}\,\frac{1}{(2\pi)^2}\,d\Omega\;\frac{\mathcal{E}_{k'}}{4}\;\frac{1}{m+\mathcal{E}_k\underbrace{(1-\cos\theta)}_{\frac{m}{\mathcal{E}_{k'}}-\frac{m}{\mathcal{E}_k}}}\;\langle |M|^2\rangle_{Lab}$$

$$= \frac{1}{4m\mathcal{E}_k}\,\frac{1}{(2\pi)^2}\,d\Omega\;\frac{\mathcal{E}_{k'}^2}{4\mathcal{E}_k m}\;\langle |M|^2\rangle_{Lab}$$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{d\cos\theta}\right)_{Lab} = \frac{1}{32m^2\pi}\left(\frac{\mathcal{E}_{k'}}{\mathcal{E}_k}\right)^2\langle |M|^2\rangle_{Lab}}$$

Plugging in the amplitude squared we get

$$\boxed{\left(\frac{d\sigma}{d\cos\theta}\right)_{Lab} = \frac{\pi\alpha_{EM}^2}{m^2}\left(\frac{\varepsilon_{k'}}{\varepsilon_k}\right)^2\left[\frac{\varepsilon_k}{\varepsilon_{k'}} + \frac{\varepsilon_{k'}}{\varepsilon_k} - \sin^2\theta\right]}$$

Klein-Nishina formula (1929).

Low energy limit $\varepsilon_k \to 0 \Rightarrow \varepsilon_{k'} \simeq \varepsilon_k \Rightarrow$

$$\left(\frac{d\sigma}{d\cos\theta}\right)_{Lab} \approx \frac{\pi\alpha_{EM}^2}{m^2}\left[1+\cos^2\theta\right]$$

$$\sigma_{tot}^{Lab} \approx \frac{8}{3}\frac{\pi\alpha_{EM}^2}{m^2}$$

Thomson scattering x-section:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha_{EM}^2}{m^2}(\varepsilon_1\cdot\varepsilon_2)^2$$

polarizations before & after.

Also,

$$\frac{d\sigma^{e^-\gamma\to e^-\gamma}}{dt} = \frac{1}{(4\pi)^2}\frac{\pi}{s(s-m_e^2)}\langle|M|^2\rangle$$

$$\Rightarrow \frac{d\sigma^{e^-\gamma\to e^-\gamma}}{dt} = \frac{4\pi\alpha_{EM}^2}{s(s-m^2)}\left\{-\frac{1}{2}\left(\frac{u-m^2}{s-m^2}+\frac{s-m^2}{u-m^2}\right) + 2m^2\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right) + 2m^4\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right)^2\right\}.$$

# The Optical Theorem and Cutkosky Rules.

Start from the $S$-matrix : $S^+ S = \mathbb{1}$. (Unitarity).

$$S = \mathbb{1} + i\mathbb{T} \Rightarrow (\mathbb{1} - iT^+)(1 + iT) = 1$$

$$\Rightarrow \quad i(T - T^+) + T^+ T = 0$$

$$\Rightarrow \boxed{-i(T - T^+) = T^+ T}$$ unitarity condition for $T$-matrix.

Sandwitch all this between states $|k_1, k_2\rangle$ :

$$-i\langle k_1', k_2' | T - T^+ | k_1, k_2\rangle = \langle k_1', k_2' | T^+ T | k_1, k_2\rangle.$$

as $\langle p_1, \ldots, p_n | \; T | k_1, k_2\rangle = (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_i p_i) \, M_{2\to n}$

$$\Rightarrow \text{lhs} = \left[-iM(k_1, k_2 \to k_1', k_2') + iM^*(k_1', k_2' \to k_1, k_2)\right](2\pi)^4 \delta^{(4)}(k_1 + k_2 - k_1' - k_2').$$

$$\text{rhs} = \sum_n \langle k_1', k_2' | T^+ \underbrace{|n\rangle\langle n|}_{\text{complete set of states}} T | k_1, k_2\rangle =$$

$$= \sum_n \int \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3 2E_{q_i}} \langle k_1', k_2' | T^+ | q_1 \ldots q_n\rangle\langle q_1 \ldots q_n | T | k_1, k_2\rangle.$$

$$= \sum_n \int \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3 2E_{q_i}} \; M(k_1, k_2 \to q_i) \, M^*(k_1', k_2' \to q_i) \cdot (2\pi)^4 \cdot$$

$$\cdot \delta^{(4)}(k_1 + k_2 - \sum_i q_i) \, (2\pi)^4 \, \delta(k_1 + k_2 - k_1' - k_2').$$

Equating lhs = rhs & dropping δ-function yields:

$$-i\left[ M(k_1,k_2 \to k_1,k_2) - M^*(k_1,k_2 \to k_1,k_2) \right] = \sum_n \int \prod_{i=1}^{n} \frac{d^3 q_i}{(2\pi)^3 2\mathcal{E}_{q_i}} \cdot$$

$$\cdot \left| M(k_1,k_2 \to \{q_i\}) \right|^2 (2\pi)^4 \delta^{(4)}\left(k_1+k_2 - \sum_i q_i\right)$$

where we replaced $k_1', k_2' \to k_1, k_2$.

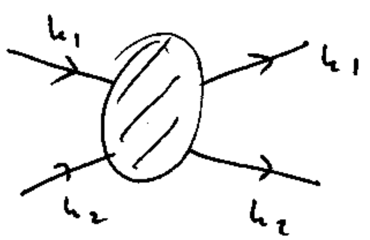
Right-hand side now looks just like $2 \to n$ cross section summed over all $n$ => it is

$\sigma_{tot} \cdot 2\mathcal{E}_{k_1} \, 2\mathcal{E}_{k_2} \, |\vec{v}_1 - \vec{v}_2|$. We write

$$\boxed{\sigma_{tot} = \frac{1}{2\mathcal{E}_{k_1} \, 2\mathcal{E}_{k_2} \, |\vec{v}_1 - \vec{v}_2|} \cdot 2 \, \mathrm{Im} \, M(k_1,k_2 \to k_1,k_2)}$$

Optical Theorem
(cf. E&M, QM)

$M(k_1,k_2 \to k_1,k_2)$ ~ forward scattering amplitude
(final state = initial state)




Optical Th'm is very useful, true for ∀ # of external legs.

The cross section is the amplitude squared.

Let us represent it diagrammatically as: