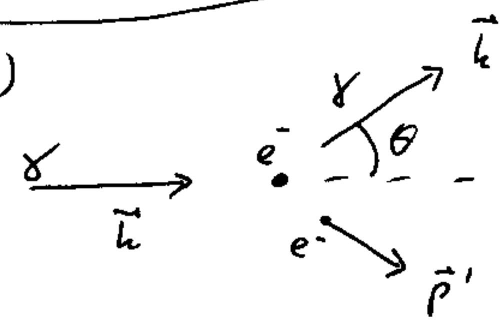
<u>Correction</u>: sign fixing in Feynman rules for fermions: associate (-1) for

- each fermion line that begins & ends in the initial state (drop final state from this rule)
- each fermion loop (still the same as before).

<u>Last time</u> finished calculating the x-section for Compton scattering: $e^-\gamma \to e^-\gamma$. In electron's rest frame we got

$$\left(\frac{d\sigma}{d\cos\theta}\right)_{Lab} = \frac{\pi \alpha_{EM}^2}{m^2}\left(\frac{\mathcal{E}_{k'}}{\mathcal{E}_k}\right)^2\left[\frac{\mathcal{E}_k}{\mathcal{E}_{k'}} + \frac{\mathcal{E}_{k'}}{\mathcal{E}_k} - \sin^2\theta\right]$$

Klein-Nishina f-la (1929)



Started talking about

The Optical Theorem and Cutkosky Rules (cont'd)

S - matrix: $S^+ S = \mathbb{1}$ (unitarity)

$$S = \mathbb{1} + i\,\mathbb{T} \Rightarrow -i\left[\mathbb{T} - \mathbb{T}^+\right] = \mathbb{T}^+\mathbb{T}$$

S, T - operators.

Plugging in the amplitude squared we get

$$\boxed{\left(\frac{d\sigma}{d\cos\theta}\right)_{Lab} = \frac{\pi\alpha_{EM}^2}{m^2}\left(\frac{\varepsilon_{\omega'}}{\varepsilon_\omega}\right)^2\left[\frac{\varepsilon_\omega}{\varepsilon_{\omega'}} + \frac{\varepsilon_{\omega'}}{\varepsilon_\omega} - \sin^2\theta\right]}$$

Klein-Nishina formula (1929).

Low energy limit $\varepsilon_\omega \to 0 \Rightarrow \varepsilon_{\omega'} \simeq \varepsilon_\omega \Rightarrow$

$$\boxed{\left(\frac{d\sigma}{d\cos\theta}\right)_{Lab} \approx \frac{\pi\alpha_{EM}^2}{m^2}\left[1+\cos^2\theta\right]}$$

$$\boxed{\sigma_{tot}^{Lab} \approx \frac{8}{3}\frac{\pi\alpha_{EM}^2}{m^2}}$$

Thomson scattering x-section:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha_{EM}^2}{m^2}(\varepsilon_1\cdot\varepsilon_2)^2$$

polarizations before & after.

Also,

$$\frac{d\sigma^{e^-\gamma\to e^-\gamma}}{dt} = \frac{1}{(4\pi)^2}\frac{\pi}{s(s-m_e^2)}\langle|M|^2\rangle$$

$$\Rightarrow \boxed{\frac{d\sigma^{e^-\gamma\to e^-\gamma}}{dt} = \frac{4\pi\alpha_{EM}^2}{s(s-m^2)}\left\{-\frac{1}{2}\left(\frac{u-m^2}{s-m^2}+\frac{s-m^2}{u-m^2}\right)+2m^2\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right) + 2m^4\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right)^2\right\}}.$$

The Optical Theorem and Cutkosky Rules.

Start from the S-matrix: $S^+ S = \mathbb{1}$. (Unitarity).

$$S = \mathbb{1} + i\mathbb{T} \Rightarrow \left(\mathbb{1} - iT^+\right)\left(1 + iT\right) = 1$$

$$\Rightarrow \quad i\left(T - T^+\right) + T^+ T = 0$$

$$\Rightarrow \boxed{-i\left(T - T^+\right) = T^+ T}$$ unitarity condition for T-matrix.

Sandwich all this between states $|k_1, k_2\rangle$:

$$-i\langle k_1', k_2' | T - T^+ | k_1, k_2\rangle = \langle k_1', k_2' | T^+ T | k_1, k_2\rangle.$$

as $\langle p_1, \dots, p_n | \; T | k_1, k_2\rangle = (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_i p_i)\, M_{2\to n}$

$$\Rightarrow \text{lhs} = \left[-iM(k_1, k_2 \to k_1', k_2') + iM^*(k_1', k_2' \to k_1, k_2)\right](2\pi)^4 \delta^{(4)}(k_1 + k_2 - k_1' - k_2').$$

$$\text{rhs} = \sum_n \langle k_1', k_2' | T^+ \underbrace{|n\rangle\langle n|}_{\text{complete set of states}} T | k_1, k_2\rangle =$$

$$= \sum_n \int \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3 2E_{q_i}} \langle k_1', k_2' | T^+ | q_1 \dots q_n\rangle\langle q_1 \dots q_n | T | k_1, k_2\rangle.$$

$$= \sum_n \int \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3 2E_{q_i}} \; M(k_1, k_2 \to q_i)\, M^*(k_1', k_2' \to q_i) \cdot (2\pi)^4 \cdot$$

$$\cdot \delta^{(4)}(k_1 + k_2 - \sum_i q_i)\,(2\pi)^4 \delta(k_1 + k_2 - k_1' - k_2').$$

Equating lhs = rhs & dropping δ-function yields:

$$-i\left[\mathcal{M}(k_1,k_2\to k_1,k_2)-\mathcal{M}^*(k_1,k_2\to k_1,k_2)\right]=\sum_n\int\prod_{i=1}^{n}\frac{d^3q_i}{(2\pi)^3\,2\mathcal{E}_{q_i}}\cdot$$

$$\cdot\left|\mathcal{M}(k_1,k_2\to\{q_i\})\right|^2\,(2\pi)^4\,\delta^{(4)}\Big(k_1+k_2-\sum_i q_i\Big)$$

where we replaced $k_1', k_2' \to k_1, k_2$.

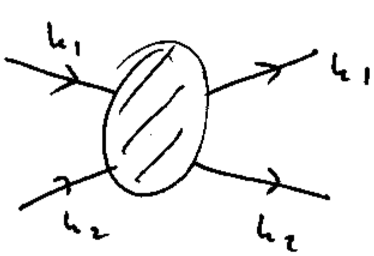
Right-hand side now looks just like $2\to n$ cross section summed over all n ⇒ it is

$\sigma_{tot}\cdot 2\mathcal{E}_{k_1}\,2\mathcal{E}_{k_2}\,|\vec{v}_1-\vec{v}_2|$. We write

$$\boxed{\sigma_{tot}=\frac{1}{2\mathcal{E}_{k_1}\,2\mathcal{E}_{k_2}\,|\vec{v}_1-\vec{v}_2|}\cdot 2\,\mathrm{Im}\,\mathcal{M}(k_1,k_2\to k_1,k_2)}$$

Optical Theorem
(cf. E&M, QM)

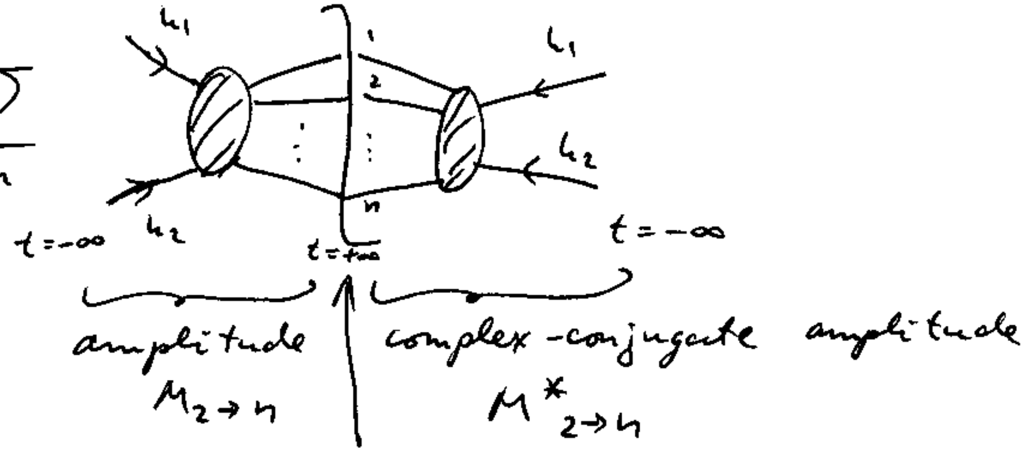
$\mathcal{M}(k_1,k_2\to k_1,k_2)$ ~ forward scattering amplitude
(final state = initial state)



Optical Th'm is very useful, true for ∀ # of external legs.

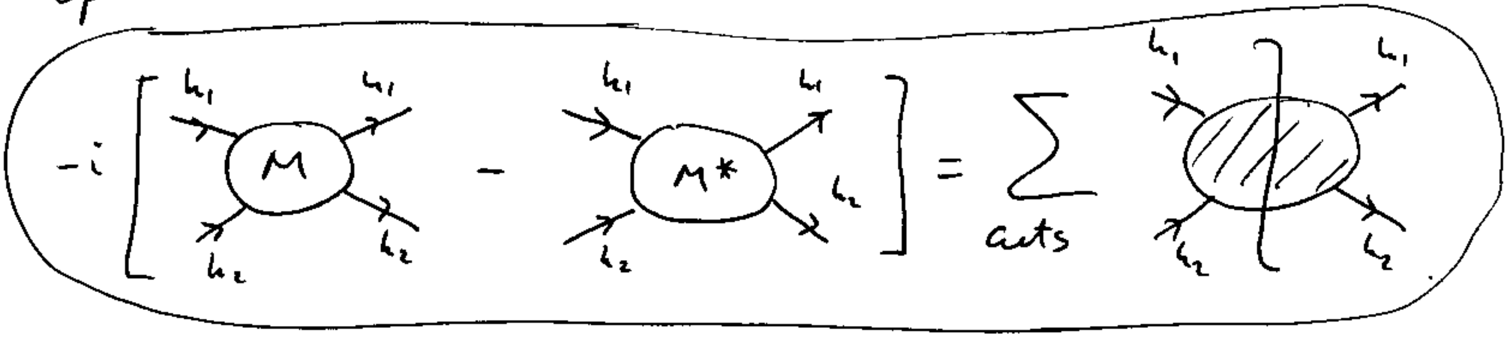
The cross section is the amplitude squared.

Let us represent it diagrammatically as:

$\sigma_{total} = \sum_n$ 

"Cut" denotes the final state.

Optical theorem can be drawn as



This is the essence of <u>Cutkosky rules</u>: Im part of a diagram is given by the sum over all the cuts.

<u>Cutkosky rules</u>: to get $2i \, \mathrm{Im}\, M$:

(i) Cut in all possible ways.

(ii) In each cut, replace cut propagators' denominators:

$$\frac{1}{p^2-m^2+i\varepsilon} \to -2\pi i\, \delta(p^2-m^2)$$

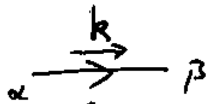
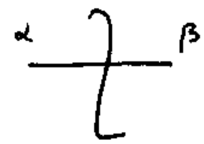
(iii) Sum the contributions of all cuts.

$$2\,\mathrm{Im}\,\frac{1}{p^2-m^2+i\epsilon} = -2\pi i\,\delta(p^2-m^2) \Rightarrow \text{ a cut scalar propagator}$$

gives  $\longrightarrow$ 

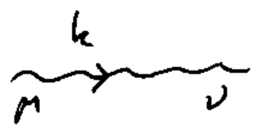
$$\frac{i}{k^2-m^2+i\epsilon} \qquad i\cdot(-2\pi i)\,\delta(k^2-m^2) = 2\pi\,\delta(k^2-m^2)$$

puts particle on mass shell, as required

Fermions:  $\longrightarrow$ 

$$\frac{i(\gamma\cdot k+m)_{\beta\alpha}}{k^2-m^2+i\epsilon} \qquad i(\not{k}+m)_{\beta\alpha}(-2\pi i)\,\delta(k^2-m^2) =$$

$$= \underbrace{(\not{k}+m)_{\beta\alpha}}_{\sum_r u_r(k)_\beta \bar{u}_r(k)_\alpha} 2\pi\,\delta(k^2-m^2)$$

Gauge bosons:  $\longrightarrow$ 

$$\frac{-ig_{\mu\nu}}{k^2+i\epsilon} \qquad -ig_{\mu\nu}\,(-2\pi i)\,\delta(k^2) =$$

$$= \underbrace{-\,g_{\mu\nu}}\,(2\pi)\,\delta(k^2)$$

usual replacement $\longrightarrow \sum_{\lambda=\pm} \varepsilon_\mu^{(\lambda)*}(k)\,\varepsilon_\nu^{(\lambda)}(k).$