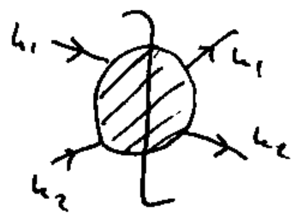
Last time Finished the discussion of the Optical Theorem:

$$\boxed{\sigma_{tot} = \frac{1}{2\mathcal{E}_{k_1}\, 2\mathcal{E}_{k_2}\, |\vec{v}_1 - \vec{v}_2|} \cdot 2\,\mathrm{Im}\,\mathcal{M}(k_1,k_2 \to k_1,k_2)}$$

Formulated Cutkosky rules:

$$\mathrm{Im}\,\mathcal{M} = \sum_{\text{cuts}}$$



(true for $\forall$ number of external legs, spinors, gauge bosons...)

for each cut propagator replace: $\frac{1}{p^2-m^2+i\varepsilon} \to -2\pi i\,\delta(p^2-m^2)$.

Regularization & Renormalization (cont'd)

Field-Strength Renormalization: the Electron Self-Energy. (cont'd)

Dirac field ψ:

$$S(p) \equiv \int d^4x\, e^{ip\cdot x} \langle \Psi_0 | T\, \psi(x)\bar{\psi}(0) | \Psi_0\rangle = Z_2 \frac{i}{\not{p} - m_{phys}} + \begin{pmatrix}\text{multi-particle}\\ \text{contributions}\end{pmatrix}$$

Z_2: field-strength renormalization

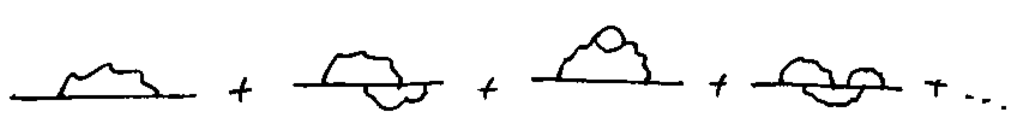
m_{phys}: physical mass

Want to sum:

(Def.) 1PI diagrams: can not be split in two by removing one line.

(Def.) $-i\Sigma(p)$ = sum of 1PI corrections to electron propagator

$\equiv$

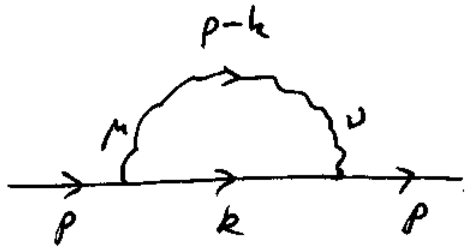


Dressed propagator is equal to:

$$\boxed{S(p) = \frac{i}{\not{p} - m_0 - \Sigma(p)}}$$

"bare" mass

Let's calculate $\Sigma(p)$ perturbatively:



$$= \frac{i}{\not{p} - m_0}\left[-i\,\Sigma_2(p)\right]\frac{i}{\not{p} - m_0}$$

$$\boxed{-i\,\Sigma_2(p) = -e^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{-2\not{k} + 4m_0}{k^2 - m_0^2 + i\varepsilon}\, \frac{1}{(p-k)^2 + i\varepsilon}}$$ (after some algebra).

Superficially the divergence is $\Sigma_2 \sim \int^{\Lambda} dk \sim \Lambda \sim$ linear,

just like Coulomb divergence:

e $\quad |\vec{E}| = \frac{e}{r^2} \Rightarrow$ energy $\sim \int d^3r\, \vec{E}^2 \sim e^2 \int \frac{d^3r}{r^4}$

$$\sim e^2 \int_{1/\Lambda} \frac{dr}{r^2} \sim \Lambda \sim \text{linear too.}$$

The diagram is

$$\int \frac{i}{\not p - m_0} (-ie\gamma^\nu) \frac{i(\not k + m_0)}{k^2 - m_0^2 + i\varepsilon} (-ie\gamma^\mu) \frac{i}{\not p - m_0} \frac{d^4k}{(2\pi)^4} \cdot \frac{-ig_{\mu\nu}}{(p-k)^2 + i\varepsilon}$$

need $i\varepsilon$ to integrate.

$$\Rightarrow -i\Sigma_2(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{\not k + m_0}{k^2 - m_0^2 + i\varepsilon} \gamma_\mu \frac{1}{(p-k)^2 + i\varepsilon}.$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-2\not k + 4m_0}{k^2 - m_0^2 + i\varepsilon} \frac{1}{(p-k)^2 + i\varepsilon}.$$

$\Rightarrow$ Note that the answer has the structure: $f_1(p^2)\not p + f_2(p_2)$.

$\Rightarrow$ use <u>Feynman parameters</u>:

$$\boxed{\frac{1}{A_1 A_2 \ldots A_n} = \int_0^1 dx_1 \ldots dx_n \, \delta\Big(\sum_{i=1}^n x_i - 1\Big) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \ldots + x_n A_n]^n}}$$

induction.

which for $n=2$ reduces to

check

$$\frac{1}{AB} = \int_0^1 dx\,dy\, \delta(1-x-y) \frac{1}{[xA + yB]^2} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}$$

In our case $B = k^2 - m_0^2 + i\varepsilon$, $A = (p-k)^2 + i\varepsilon$

$$\Rightarrow -i\Sigma_2 = -e^2 \int \frac{d^4k}{(2\pi)^4} \cdot (-2\not k + 4m_0) \int_0^1 dx \frac{1}{\big[x[(p-k)^2 + i\varepsilon] + (1-x)[k^2 - m_0^2 + i\varepsilon]\big]^2} = -e^2 \int \frac{d^4k}{(2\pi)^4} (-2\not k + 4m_0) \cdot \int_0^1 dx \,\cdot$$

$$\cdot \frac{1}{\left[k^2 - 2x p\cdot k + x p^2 - (1-x) m_0^2 + i\varepsilon\right]^2} .$$

Problem: the integral is divergent at large-k (ultraviolet-, or, UV-divergent): $\int^{\Lambda} \frac{d^4k}{k^4}(\not k + \text{const}) \sim \ln \Lambda$.

Def. Let us regulate the divergence by subtracting a massive photon propagator from the photon propagator:

$$\frac{1}{(p-k)^2 + i\varepsilon} \to \frac{1}{(p-k)^2 + i\varepsilon} - \frac{1}{(p-k)^2 - M^2 + i\varepsilon} .$$

As we take $M \to \infty$ the second term disappears. For large-M can explicitly see the divergence regularization. This is Pauli-Villars regularization!

(Subtraction of other heavy particles, such as scalars or fermions, is also possible.)

$$-i\Sigma_2^{reg}(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} (-2\not k + 4m_0) \int_0^1 dx \left\{ \frac{1}{\left[k^2 - 2xp\cdot k + xp^2 - (1-x)m_0^2 + i\varepsilon\right]^2} \right.$$

$$\left. - \frac{1}{\left[k^2 - 2xp\cdot k + xp^2 - (1-x)m_0^2 - xM^2 + i\varepsilon\right]^2} \right\} = \left| \begin{array}{l} \text{define} \\ \tilde{k}^\mu = k^\mu - xp^\mu \\ \text{\& drop tildes} \end{array} \right. =$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} (-2\overset{-2x\not p}{\check{\not k}} + 4m_0) \int_0^1 dx \left\{ \frac{1}{\left[k^2 + x(1-x)p^2 - (1-x)m_0^2 + i\varepsilon\right]^2} \right. -$$

$$\left. - \frac{1}{\left[k^2 + x(1-x)p^2 - (1-x)m_0^2 - xM^2 + i\varepsilon\right]^2} \right\}$$

We can now drop the term linear in k from the numerator, as it is zero. We have

$$\Sigma_2^{reg}(p) = -i e^2 \int_0^1 dx\, (-2x\not{p} + 4m_0) \int \frac{d^4k}{(2\pi)^4} \cdot \left\{ \frac{1}{[k^2 + x(1-x)p^2 - (1-x)m_0^2 + i\varepsilon]^2} - \frac{1}{[k^2 + x(1-x)p^2 - (1-x)m_0^2 - xM^2 + i\varepsilon]^2} \right\}.$$

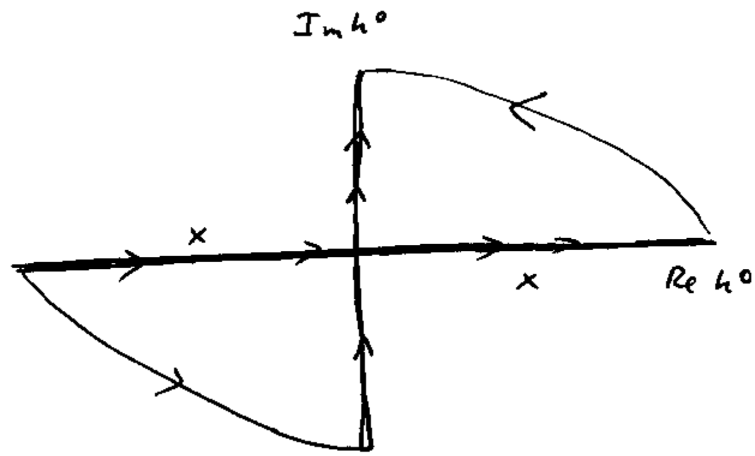
To integrate over d^4k need to perform <u>Wick rotation</u>.

Consider an integral:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - \Lambda^2 + i\varepsilon]^2}$$

with $\Lambda^2 > 0$. The k^0-integral is

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \cdot \frac{1}{\left(k^0 - \sqrt{\vec{k}^2 + \Lambda^2} + i\varepsilon\right)^2 \left(k^0 + \sqrt{\vec{k}^2 + \Lambda^2} - i\varepsilon\right)^2}.$$



The integration contour could be distorted as shown above, the quarter-circles at ∞ can be dropped. We are left with the integral along Im axis.

Write $k^0 = i k^0_E$. $\Rightarrow$ $\boxed{d^4k = i\, d^4k_E}$, $k^2 = -k_E^2$

where $k_E^2 = -(k^0_E)^2 - \vec{k}^2$.

E = Euclidean space (though 4-dim).

The integral becomes:

$$: \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{(k_E^2+\Lambda^2)^2}$$

. Now we can separately integrate over angles and over absolute value of k_E:

$$d^4 k_E = d\Omega_4 \, dk_E \cdot k_E^3 .$$

$\int d\Omega_4 = 2\pi^2$ ~ surface area of a sphere in 4 dim.

$$\boxed{\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$ in general.

$$x = r(\sin\omega \sin\theta \cos\varphi, \sin\omega \sin\theta \sin\varphi, \sin\omega \cos\theta, \cos\omega)$$

$d^4 x = r^3 \sin^2\omega \sin\theta \, d\varphi d\theta d\omega dr$ ⇒ can calculate the area of the 4-dim sphere.

⇒ the integral now is $i \, 2\pi^2 \int_0^\infty \frac{dk_E \cdot k_E^3}{(2\pi)^4} \frac{1}{(k_E^2+\Lambda^2)^2} =$

$= \frac{i}{16\pi^2} \int_0^\infty dk_E^2 \cdot \frac{k_E^2}{(k_E^2+\Lambda^2)^2}$. We have

$$\Sigma_2^{reg}(p) = \frac{e^2}{16\pi^2} \int_0^1 dx(-2x\not{p} + 4m_0) \int_0^\infty dk_E^2 \cdot k_E^2 \cdot$$

$$\cdot \left\{ \frac{1}{[k_E^2 - x(1-x)p^2 + (1-x)m_0^2]^2} - \frac{1}{[k_E^2 - x(1-x)p^2 + (1-x)m_0^2 + xM^2]^2} \right\}$$

Now, $\int_0^\infty dz \cdot \left[\frac{z}{(z+a)^2} - \frac{z}{(z+b)^2} \right] = \int_0^\infty dz \left[\frac{1}{z+a} - \frac{1}{z+b} - \frac{a}{(z+a)^2} + \right.$

$$+ \frac{b}{(z+b)^2}\Big] = \ln\left(\frac{z+a}{z+b}\right)\Big|_0^\infty + \frac{a}{z+a}\Big|_0^\infty - \frac{b}{z+b}\Big|_0^\infty =$$

$$= -\ln\left(\frac{a}{b}\right) - 1 + 1 = \ln\left(\frac{b}{a}\right) \Rightarrow$$

$$\Sigma_2^{reg}(p) = \frac{e^2}{16\pi^2}\int_0^1 dx\left(-2x\not{p} + 4m_0\right) \cdot \ln\left[\frac{xM^2 - x(1-x)p^2 + (1-x)m_0^2}{-x(1-x)p^2 + (1-x)m_0^2}\right]$$

We only keep terms that diverge and/or remain finite as $M^2 \to \infty$ $\Rightarrow$

$$\boxed{\Sigma_2^{reg}(p) = \frac{\alpha_{EM}}{2\pi}\int_0^1 dx\,(2m_0 - x\not{p})\ln\left[\frac{xM^2}{(1-x)m_0^2 - x(1-x)p^2}\right].}$$

At this one-loop level the dressed propagator is

$$S(p) = \frac{i}{\not{p} - m_0 - \Sigma_2(p)} = Z_2\frac{i}{\not{p} - m_{phys}} + \text{multi-particle contributions.}$$

$\Rightarrow$ first of all the poles should coincide:

$$\Rightarrow \boxed{\left[\not{p} - m_0 - \Sigma_2(p)\right]\Big|_{\not{p} = m_{phys}} = 0}$$

$$\Rightarrow m_{phys} - m_0 = \Sigma_2(p)\Big|_{\substack{p^2 = m_{phys}^2 \\ \not{p} = m_{phys}}} = \Sigma_2(p)\Big|_{\substack{p^2 = m_0^2 \\ \not{p} = m_0}} + \underbrace{O(\alpha_{EM}^2)}_{\text{neglect, beyond our accuracy}}$$

Putting $\not{p} = m_{phys}$ is a bit of cheating that works.

Let us do the calculation correctly: note that

$$\Sigma(p) = A(p^2)\not{p} + B(p^2).$$

We want $\dfrac{i}{\not{p} - m_0 - \Sigma_2(p)} = Z_2 \dfrac{i}{\not{p} - m_{phys}} + \cdots$

$$\frac{i}{\not{p} - m_0 - A(p^2)\not{p} - B(p^2)} = \frac{i}{(1-A(p^2))\not{p} - m_0 - B(p^2)} =$$

$$= \frac{i\left[(1-A(p^2))\not{p} + m_0 + B(p^2)\right]}{[1-A(p^2)]^2 p^2 - (m_0 + B(p^2))^2} \underset{\text{want}}{\overset{\uparrow}{=}} Z_2 \frac{i(\not{p} + m_{phys})}{p^2 - m_{phys}^2} + \cdots$$

$$\Rightarrow \left\{[1 - A(p^2)]^2 p^2 - (m_0 + B(p^2))^2\right\}\Big|_{p^2 = m_{phys}^2} = 0$$

$$\Rightarrow [1 - A(m_{phys}^2)]^2 m_{phys}^2 = (m_0 + B(m_{phys}^2))^2$$

$$\Rightarrow m_{phys} = A(m_{phys}^2)m_{phys} + m_0 + B(m_{phys}^2) = m_0 + \Sigma(p)\Big|_{\not{p} = m_{phys}}$$

$$\Rightarrow \text{get } m_{phys} - m_0 = \Sigma(p)\Big|_{\not{p}=m_{phys}} \text{ just like before.}$$

Let us now expand the denominator:

$$[1 - A(p^2)]^2 p^2 - (m_0 + B(p^2))^2 = \underset{\uparrow\, p^2 = m_{phys}^2}{\emptyset} + \frac{\partial}{\partial p^2}\left[[1-A(p^2)]^2 p^2 - (m_0 + B(p^2))^2\right]\Big|_{p^2 = m_{phys}^2} \cdot$$

$$\cdot (p^2 - m_{phys}^2) + \cdots = \Big\{[1 - A(m_{phys}^2)]^2 + 2[1 - A(m_{phys}^2)](-)A'(m_{phys}^2)m_{phys}^2$$

$$-2\underbrace{\left(m_0+B(m_{phys}^2)\right)}_{\left[1-A(m_{phys}^2)\right]m_{phys}}\cdot B'(m_{phys}^2)\Big\}\left(p^2-m_{phys}^2\right)+\ldots=$$

$$=\left[1-A(m_{phys}^2)\right]\left\{1-A(m_{phys}^2)-2A'(m_{phys}^2)\cdot m_{phys}^2-2m_{phys}B'(m_{phys}^2)\right\}\cdot$$

$$\cdot\left(p^2-m_{phys}^2\right)+\ldots$$

The singular part of the propagator is:

$$\frac{i\left(\not p+m_{phys}\right)}{\left(p^2-m_{phys}^2\right)\left[1-A-2A'm_{phys}^2-2m_{phys}B'\right]}=\frac{i\left(\not p+m_{phys}\right)}{\left(p^2-m_{phys}^2\right)}\cdot Z_2$$

$$\Rightarrow\boxed{\frac{1}{Z_2}=1-A(m_{phys}^2)-2A'(m_{phys}^2)\,m_{phys}^2-2B'(m_{phys}^2)\,m_{phys}}$$

Compare with $1-\left.\dfrac{\partial\Sigma}{\partial\not p}\right|_{\not p=m_{phys}}=1-\dfrac{\partial}{\partial\not p}\left[A(\not p^2)\not p+B(\not p^2)\right]\Big|_{\not p=m_{phys}}$

$$=1-A(m_{phys}^2)-2m_{phys}^2A'(m_{phys}^2)-2m_{phys}\cdot B'(m_{phys}^2).$$

$\Rightarrow$ the "cheating" trick gives exactly the same result as the exact calculation!

$$\Rightarrow \quad \delta m \equiv m_{phys} - m_0 = \Sigma_2(p)\Big|_{p = m_0} \quad \sim \text{mass shift}$$

$$\Rightarrow \quad \delta m = \frac{\alpha_{EM}}{2\pi} m_0 \cdot \int_0^1 dx\,(2-x)\cdot \ln\left[\frac{x M^2}{(1-x)m_0^2 - x(1-x)m_0^2}\right]$$

$$\delta m = \frac{\alpha_{EM}}{2\pi} m_0 \cdot \int_0^1 dx\cdot(2-x)\cdot \ln\left[\frac{x M^2}{(1-x)^2 m_0^2}\right]$$

x-integral is doable:

$$\delta m = \frac{\alpha_{EM}}{2\pi}\cdot m_0\cdot\left\{\frac{3}{2}\ln\left(\frac{M^2}{m_0^2}\right) + \underbrace{\int_0^1 dx\cdot(2-x)\cdot\ln\frac{x}{(1-x)^2}}_{3/4}\right\}$$

$$\Rightarrow \boxed{\delta m = \frac{3\alpha_{EM}}{4\pi} m_0 \left\{\ln\left(\frac{M^2}{m_0^2}\right) + \frac{1}{2}\right\}}.$$