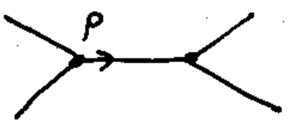
Correction Cutkosky rules as stated give $2i\,\mathrm{Im}\,M$.

Example φ^3 theory:



$$iM = (-i\lambda)^2 \frac{i}{p^2-m^2+i\varepsilon} = -i\frac{\lambda^2}{p^2-m^2+i\varepsilon} \Rightarrow$$

$$\Rightarrow M = -\frac{\lambda^2}{p^2-m^2+i\varepsilon} \Rightarrow 2i\,\mathrm{Im}\,M = 2i(-\lambda^2)\underbrace{\mathrm{Im}\frac{1}{p^2-m^2+i\varepsilon}}_{-\pi\,\delta(p^2-m^2)}$$

$$= i2\pi\,\delta(p^2-m^2)\cdot\lambda^2\,;$$

Using Cutkosky rules: $2i\,\mathrm{Im}\,M = (-i\lambda)^2\cdot\cancel{i}\cdot(-2\pi i)\cdot$

$$\cdot\,\delta(p^2-m^2)\underbrace{\cancel{(-i)}}_{\text{inverting the "}i\text{" in }iM.} = i\,2\pi\,\delta(p^2-m^2)\cdot\lambda^2.$$

Alternatively one can modify Cutkosky rules to give simply $\mathrm{Im}\,M$ by replacing rule (ii) with

$$\frac{1}{p^2-m^2+i\varepsilon} \rightarrow -\pi\,\delta(p^2-m^2).$$

Last time

Field-Strength Renormalization: the Electron Self-Energy

(cont'd)

Dressed electron propagator:

$$S(p) = \frac{i}{\not{p} - m_0 - \Sigma(p)} \underset{\substack{\uparrow \\ \text{want}}}{=} Z_2 \frac{i}{\not{p} - m_{phys}} + \cdots$$

$-i\Sigma(p)$ ~ sum of all 1PI diagrams.

We calculated  using Pauli-Villars regularization to get

$$\boxed{\Sigma_2^{reg}(p) = \frac{\alpha_{EM}}{2\pi} \int_0^1 dx \, (2m_0 - x\not{p}) \ln\left[\frac{x M^2}{(1-x)m_0^2 - x(1-x)p^2}\right]}$$

Matching the poles required $\boxed{\left[\not{p} - m_0 - \Sigma_2(p)\right]\Big|_{\not{p}=m_{phys}} = 0}$

& get $\delta m \equiv m_{phys} - m_0 = \Sigma_2(p)\Big|_{\not{p}=m_0} + O(\alpha_{EM}^2)$.

This gave the mass shift:

$$\boxed{\delta m = \frac{3\alpha_{EM}}{4\pi} m_0 \left\{ \ln\left(\frac{M^2}{m_0^2}\right) + \frac{1}{2} \right\}}$$

M ~ UV regulator.

To find the pole "residue" Z_2 expand around it:

$$\not{p} - m_0 - \Sigma_2 \underset{\substack{\uparrow \\ \text{can think of as ftn of } \not{p}}}{(\not{p})} = \not{p} \underbrace{- m_0 - \Sigma_2(\not{p} = m_{phys})}_{= -m_{phys}} - \frac{\partial \Sigma_2}{\partial \not{p}}\Big|_{\not{p} = m_{phys}} \cdot$$

$$\cdot (\not{p} - m_{phys}) + o\left((\not{p} - m_{phys})^2\right) =$$

$$= (\not{p} - m_{phys})\left(1 - \frac{\partial \Sigma_2}{\partial \not{p}}\Big|_{\not{p} = m_{phys}}\right) + \dots$$

$$\Rightarrow \boxed{\frac{1}{Z_2} = 1 - \frac{\partial \Sigma_2}{\partial \not{p}}\Big|_{\not{p} = m_{phys}}}$$

$$\Rightarrow \frac{1}{Z_2} - 1 = - \frac{\partial \Sigma_2}{\partial \not{p}}\Big|_{\not{p} = m_{phys}} = - \frac{\partial}{\partial \not{p}}\left\{ \frac{\alpha_{EM}}{2\pi} \int_0^1 dx\,(2m_0 - x\not{p}) \cdot \right.$$

$$\left. \cdot \ln\left[\frac{xM^2}{(1-x)m_0^2 - x(1-x)\underset{\substack{\uparrow \\ \text{note!}}}{\not{p}^2}}\right]\right\}\Bigg|_{\substack{\not{p} = m_{phys} \\ \approx \\ m_0 + o(\alpha_{EM})}} = \frac{\alpha_{EM}}{2\pi} \int_0^1 dx \cdot x \cdot$$

$$\cdot \left\{ \ln\left(\frac{xM^2}{(1-x)^2 m_0^2}\right) + \cancel{m_0}(2-x) \cdot \frac{-\cancel{(1-x)}\,2\cancel{m_0}}{(1-x)^{\cancel{2}}\cancel{m_0^2}} \right\} + o(\alpha_{EM}^2)$$

$\Rightarrow$ defining $\boxed{\delta Z_2 = Z_2 - 1}$ we write

$$\delta Z_2 = -\frac{\alpha_{EM}}{2\pi}\int_0^1 dx\cdot x\cdot\left\{\ln\left[\frac{xM^2}{(1-x)^2 m_0^2}\right] - \underbrace{\frac{2(2-x)}{1-x}}_{\frac{2(1+(1-x))}{1-x} = 2+\frac{2}{1-x}}\right\}$$

can do most x-integrals:

$$\Rightarrow \delta Z_2 = -\frac{\alpha_{EM}}{2\pi}\left\{\frac{1}{2}\ln\left(\frac{M^2}{m_0^2}\right)+\frac{5}{4}-1-\underbrace{\int_0^1 dx\cdot \underset{\substack{\parallel\\ x-1+1}}{x}\cdot\frac{2}{1-x}}_{+2-2\int_0^1\frac{dx}{1-x}}\right\}$$

$$\Rightarrow \delta Z_2 = -\frac{\alpha_{EM}}{2\pi}\left\{\frac{1}{2}\ln\left(\frac{M^2}{m_0^2}\right)+\underbrace{\frac{5}{4}-1+2}_{9/4}-2\int_0^1\frac{dx}{1-x}\right\}$$

$$\Rightarrow \boxed{\delta Z_2 = -\frac{\alpha_{EM}}{4\pi}\left\{\ln\left(\frac{M^2}{m_0^2}\right)+\frac{9}{2}-4\int_0^1\frac{dx}{1-x}\right\}}.$$

$\sim$ The divergence in x-integral can be removed by introducing photon mass μ. The divergence comes from small momenta $\sim$ infrared (IR) divergence (aka collinear divergence, comes from $p^\mu = k^\mu$, $p^2 = m_0^2$, no quark recoil). Such divergences can be remedied by properly defining observables (e.g. no single-quark production x-section, need IR resolution scale / cutoff).

~ Both δZ_2 and δm are UV-divergent! This is not surprising since we've calculated electron's self-energy , like electron feeling its own Coulomb field $\Rightarrow$ UV-divergent.

$\Rightarrow$ OK: δm is infinite. Does it mean that m_{phys} is infinite too? Only if m_0 is finite. But m_{phys} is the only *observable* here: demand that m_{phys} is finite (and equal to the measured electron's mass) $\Rightarrow$ m_0 would depend on the cutoff M to satisfy this requirement.

$\Rightarrow$ We'll rearrange perturbation theory to systematically replace m_0 with m.

Vacuum Polarization

Let us now perform the resummation for the photon propagator. We need to sum graphs like

As usual start with one-particle irreducible diagrams:

$$\equiv i\Pi^{\mu\nu}(q)$$

(propagators not included).

$$\Pi^{\mu\nu}(q) = A(q^2)g^{\mu\nu} + B(q^2)q^\mu q^\nu$$

on general grounds. Impose current conservation

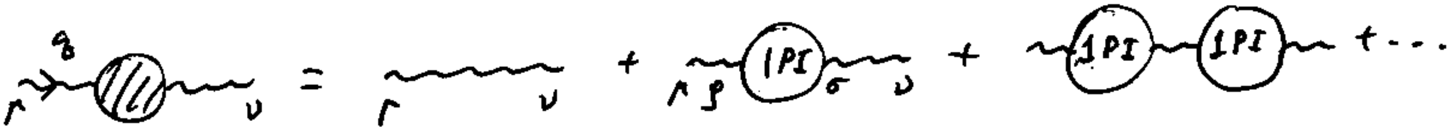
$$\Rightarrow q^\mu \Pi_{\mu\nu} = 0 \Rightarrow A q^\nu + B \cdot q^2 \cdot q^\nu = 0 \Rightarrow B = -\frac{A}{q^2}$$

$$\Rightarrow \Pi^{\mu\nu}(q) = A(q^2)\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right]$$

assume no pole in $\Pi(q^2)$ at $q^2=0$.

$$\Rightarrow \text{write } \boxed{\Pi^{\mu\nu}(q) = \left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]\Pi(q^2)}$$

Summing all 1PI bubbles get:



$$-\frac{ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\, i\left[q^2 g^{\rho\sigma} - q^\rho q^\sigma\right]\Pi(q^2)\,\frac{-ig_{\sigma\nu}}{q^2} + \dots$$

$$= \frac{-i g_{\mu\nu}}{q^2} + \frac{-i g_{\mu\rho}}{q^2} \underbrace{\left[\delta^\rho{}_\nu - \frac{q^\rho q_\nu}{q^2} \right]}_{\text{projector on direction } \perp \text{ to } q^\rho} \Pi(q^2) + \dots$$

$$\Rightarrow \left[\delta^\rho{}_\alpha - \cancel{\frac{q^\rho q_\alpha}{q^2}} \right] \cdot \left[\delta^\alpha{}_\nu - \frac{q^\alpha q_\nu}{q^2} \right] = \delta^\rho{}_\nu - \frac{q^\rho q_\nu}{q^2}$$

0 as $q_\alpha \times [\dots] = 0$,

$$\Rightarrow \text{get} \quad \frac{-i g_{\mu\nu}}{q^2} + \frac{-i g_{\mu\rho}}{q^2} \left[\delta^\rho{}_\nu - \frac{q^\rho q_\nu}{q^2} \right] \cdot \underbrace{\left(\Pi + \Pi^2 + \dots \right)}_{\frac{1}{1-\Pi} - 1}$$

$$= \frac{-i}{q^2 [1 - \Pi(q^2)]} \left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right] + \cancel{\frac{-i g_{\mu\nu}}{q^2}} - \frac{-i}{q^2} \left[\cancel{g_{\mu\nu}} - \frac{q_\mu q_\nu}{q^2} \right]$$

dressed photon prop.

$$= \boxed{\frac{-i}{q^2 (1 - \Pi(q^2))} \left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right] + \frac{-i}{q^2} \frac{q_\mu q_\nu}{q^2} = D_{\mu\nu}(q)}$$

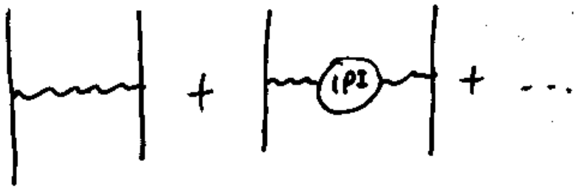
Usually couple photon propagator to some currents which are conserved ⇒ $q_\mu q_\nu$ terms are not important.

Write

$$\frac{-i g_{\mu\nu}}{q^2 (1 - \Pi(q^2))} = Z_3 \frac{-i g_{\mu\nu}}{q^2} + \left(\begin{matrix} \text{multi-particle} \\ \text{states} \end{matrix} \right)$$

$$\Rightarrow \boxed{Z_3 = \frac{1}{1 - \Pi(q^2 = 0)}}$$

(One can prove that there is no mass shift.)



$$\sim \frac{e_0^2 \, Z_3}{q^2}$$

$\Rightarrow$ can absorb photon field renormalization Z_3 into the coupling constant $\Rightarrow$

Def. Physical charge $\boxed{e^2 \equiv e_0^2 \, Z_3}$, $e = e_0 \sqrt{Z_3}$.

One also has running coupling: $\alpha_{EM}(q^2) = \frac{e^2(q^2)}{4\pi}$.

$\alpha_0 = \frac{e_0^2}{4\pi}$ $\Rightarrow$ in general get $\frac{e_0^2/4\pi}{q^2\left(1-\Pi(q^2)\right)} = \frac{\alpha_0}{q^2\left(1-\Pi(q^2)\right)}$

$$= \frac{\overset{= e^2/4\pi}{\alpha}}{q^2\left[1-\Pi(q^2)+\Pi(0)\right]} \equiv \frac{\alpha(q^2)}{q^2}$$

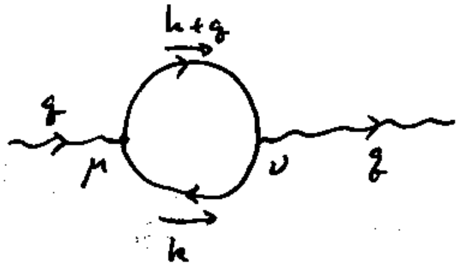
$$\Rightarrow \boxed{\alpha(q^2) = \frac{\alpha}{1-\left[\Pi(q^2)-\Pi(0)\right]}}$$

running coupling constant (q^2- dependent)

Let us calculate $\Pi_{\mu\nu}(q)$ in perturbation theory:

$$i\,\underset{\uparrow \text{order-}e^2}{\Pi_2^{\mu\nu}}(q) = (-ie)^2 \overset{\swarrow \text{fermion loop}}{(-1)} \int \frac{d^4k}{(2\pi)^4} \cdot$$

$$\mathrm{Tr}\left[\gamma^\mu \frac{i}{\not{k}-m} \gamma^\nu \frac{i}{\not{k}+\not{q}-m}\right] =$$



$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}+\not{q}+m)\right]}{\left(k^2-m^2+i\varepsilon\right)\left((k+q)^2-m^2+i\varepsilon\right)}$$