Last time | Finished calculating electron's mass shift and Z_2:

$$\boxed{\delta m = \frac{3\alpha_{EM}}{4\pi} m_0 \left\{ \ln\left(\frac{M^2}{m_0^2}\right) + \frac{1}{2} \right\} \qquad \delta Z_2 = -\frac{\alpha_{EM}}{4\pi} \left\{ \ln\left(\frac{M^2}{m_0^2}\right) + \frac{9}{2} - 4\int_0^1 \frac{dx}{1-x} \right\}}$$

IR (collinear) singularity.

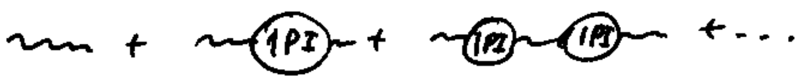
Vacuum Polarization (cont'd)

$= i\,\Pi^{\mu\nu}(q)$

Imposing $q_\mu \Pi^{\mu\nu} = 0$ wrote $\boxed{\Pi^{\mu\nu}(q) = [q^2 g^{\mu\nu} - q^\mu q^\nu]\,\Pi(q^2)}$

assume no pole at $q^2=0$.

Resumming



we got the full dressed photon propagator:

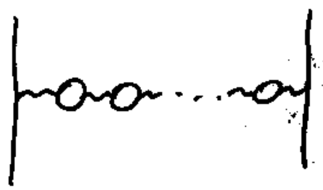
$$\boxed{D_{\mu\nu}(q) = \frac{-i}{q^2[1-\Pi(q^2)]}\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] + \frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2}}$$

$q_\mu q_\nu$ terms usually vanish $\Rightarrow$

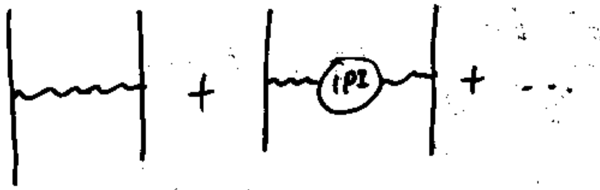
$$\frac{-i\,g_{\mu\nu}}{q^2[1-\Pi(q^2)]} \overset{\text{want}}{=} Z_3 \frac{-i g_{\mu\nu}}{q^2} + \ldots \Rightarrow \boxed{Z_3 = \frac{1}{1-\Pi(q^2=0)}}.$$

Def. Physical charge $\boxed{e \equiv e_0\sqrt{Z_3}}$.

We also defined the "effective" coupling $\alpha_{eff}(q^2)$ by requiring that e^+e^- potential is $\tilde{V}(q) \sim \frac{\alpha_{eff}(q^2)}{q^2}$:

 $\Rightarrow$ got $\alpha(q^2) = \dfrac{\alpha}{1-[\Pi(q^2)-\Pi(0)]}$

with $\alpha = \frac{e^2}{4\pi} = Z_3 \frac{e_0^2}{4\pi} = Z_3 \alpha_0$ the physical coupling (α_0 ~ bare coupling).

 $\sim \dfrac{e_0^2\, Z_3}{q^2}$

$\Rightarrow$ can absorb photon field renormalization Z_3 into the coupling constant $\Rightarrow$

Def. Physical charge $\boxed{e^2 = e_0^2\, Z_3}$, $e = e_0\sqrt{Z_3}$.

One also has running coupling: $\alpha_{EM}(q^2) = \dfrac{e^2(q^2)}{4\pi}$.

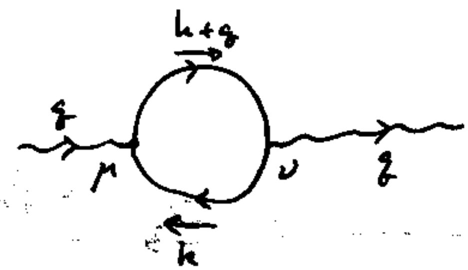
$\alpha_0 = \dfrac{e_0^2}{4\pi}$ $\Rightarrow$ in general get $\dfrac{e_0^2/4\pi}{q^2\left(1-\Pi(q^2)\right)} = \dfrac{\alpha_0}{q^2\left(1-\Pi(q^2)\right)}$

$$= \frac{\overset{=e^2/4\pi}{\alpha}}{q^2\left[1-\Pi(q^2)+\Pi(0)\right]} \equiv \frac{\alpha(q^2)}{q^2}$$

$$\Rightarrow \boxed{\alpha(q^2) = \frac{\alpha}{1-\left[\Pi(q^2)-\Pi(0)\right]}}$$ running coupling constant (q^2-dependent)

Let us calculate $\Pi_{\mu\nu}(q)$ in perturbation theory:

$$i\,\underset{\text{order-}e^2}{\Pi_2^{\mu\nu}}(q) = (-ie)^2 \underset{\text{fermion loop}}{(-1)} \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left[\gamma^\mu \frac{i}{\not k - m}\, \gamma^\nu \frac{i}{\not k + \not q - m}\right] =$$



$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu(\not k + m)\gamma^\nu(\not k + \not q + m)\right]}{(k^2-m^2+i\varepsilon)\left((k+q)^2-m^2+i\varepsilon\right)}$$

$$\Pi_2^{\mu\nu}(q) = ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu \not{k} \gamma^\nu (\not{k}+\not{q})\right] + 4m^2 g^{\mu\nu}}{(k^2-m^2+i\varepsilon)\left((k+q)^2-m^2+i\varepsilon\right)}$$

$$= 4ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{-g^{\mu\nu} k\cdot(k+q) + k^\mu (k+q)^\nu + k^\nu (k+q)^\mu + m^2 g^{\mu\nu}}{(k^2-m^2+i\varepsilon)\left((k+q)^2-m^2+i\varepsilon\right)} =$$

$$= 4ie^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu (k+q)^\mu - g^{\mu\nu}\left[k\cdot(k+q) - m^2\right]}{\left[(1-x)(k^2-m^2+i\varepsilon) + x\left((k+q)^2-m^2+i\varepsilon\right)\right]^2} =$$

$$= 4ie^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu (k+q)^\mu - g^{\mu\nu}\left[k\cdot(k+q) - m^2\right]}{\left[k^2 + 2x k\cdot q + x q^2 - m^2 + i\varepsilon\right]^2}$$

$= \Big|\, k^\mu \to k^\mu + x q^\mu \equiv \ell^\mu$ (warning: illegal operation) divergent integral $\Big|$

$$= 4ie^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell^\mu - x q^\mu)(\ell^\nu + (1-x)q^\nu) + (\ell^\nu - x q^\nu)(\ell^\mu + (1-x)q^\mu) - g^{\mu\nu}\left[(\ell - xq)\cdot(\ell+(1-x)q) - m^2\right]}{\left[\ell^2 + x(1-x)q^2 - m^2 + i\varepsilon\right]^2}$$

Numerator: drop terms linear in ℓ^μ, as $\ell^\mu \to -\ell^\mu$ demonstrates that those are zero. Remaining terms in the numerator are:

$$2\ell^\mu\ell^\nu - 2x(1-x) q^\mu q^\nu - g^{\mu\nu}\left[\ell^2 - x(1-x)q^2 - m^2\right]$$

Hence
$$\Pi_2^{\mu\nu}(q) = 4ie^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{2\ell^\mu\ell^\nu - 2x(1-x)q^\mu q^\nu - g^{\mu\nu}\left[\ell^2 - x(1-x)q^2 - m^2\right]}{\left[\ell^2 + x(1-x)q^2 - m^2 + i\varepsilon\right]^2}$$

$\Rightarrow$ problem: leading divergence $\sim g_{\mu\nu} \int^{\Lambda^2} \frac{d^4\ell}{\ell^2} \sim g_{\mu\nu}\Lambda^2$

$\Rightarrow$ no $q^2 g^{\mu\nu} - q^\mu q^\nu$ structure! why? illegal momentum shift above!

Assume that photon is space-like, $q^2<0$, & do a Wick rotation: $\ell^0 = i\ell^0_E$. We get

$$\Pi_2^{\mu\nu}(q) = -4e^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4} \frac{2\ell^\mu\ell^\nu - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}[\ell_E^2 + x(1-x)q^2 + m^2]}{[\ell_E^2 - x(1-x)q^2 + m^2]^2}.$$

Now need to regulate the integral:

- Pauli-Villars method is complicated (need to introduce 2 new massive bosons giving the loop ~ pain)
- use a different method called <u>dimensional regularization</u> ('t Hooft, Veltman, '72).

<u>Dimensional regularization</u>: replace $4 \to d$ dimensions. Calculate the integral for some general d (at small d there is no UV divergence). Take $d\to 4$ limit of the answer.

$$d^4\ell_E \to d^d\ell_E = d\ell_E \cdot \ell_E^{d-1} \cdot \underbrace{d\Omega_d}_{\text{angular integral}}$$

To find $\int d\Omega_d$ use the trick: here $\vec{x} = (x^1, x^2, \dots, x^d)$

$$(\sqrt{\pi})^d = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)^d = \int d^d x\, e^{-\vec{x}^2} = \int d\Omega_d \cdot \int_0^\infty dx \cdot x^{d-1} e^{-x^2}$$

$$= \left(\int d\Omega_d\right) \underbrace{\frac{1}{2}\int_0^\infty dx^2\, (x^2)^{\frac{d-2}{2}}\, e^{-x^2}}_{\Gamma\left(\frac{d}{2}\right)} = \int d\Omega_d \cdot \frac{1}{2}\Gamma\left(\frac{d}{2}\right)$$

as $\Gamma(z) = \int_0^\infty dt \cdot t^{z-1} e^{-t}$

$$\boxed{\int d\Omega_d = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}}$$

. See e.g. $d = 2 \Rightarrow$ get 2π

$d = 3 \Rightarrow \frac{2\pi^{3/2}}{\Gamma(3/2)} = \frac{2\pi^{3/2}}{\frac{1}{2}\underbrace{\Gamma(\frac{1}{2})}_{\sqrt{\pi}}} = 4\pi$ ok.

In d-dimensions one has $g_{\mu\nu}g^{\mu\nu} = d$. Hence we can replace:

$$\boxed{\ell^\mu \ell^\nu \to \frac{1}{d}\ell^2 g^{\mu\nu} = -\frac{1}{d}\ell_E^2 g^{\mu\nu}}$$

Minkowski

in the integral.

We have two types of integrals:

$$(i)\ \int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{\left[\ell_E^2+\Lambda^2\right]^2} = \int\frac{d\Omega_d}{(2\pi)^d}\cdot\int_0^\infty\frac{d\ell_E\cdot\ell_E^{d-1}}{\left[\ell_E^2+\Lambda^2\right]^2} = \frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)}.$$

$$\int_0^\infty d\ell_E\,\frac{\ell_E^{d-1}}{\left[\ell_E^2+\Lambda^2\right]^2} = \frac{1}{2^{d-1}\pi^{d/2}\Gamma(\frac{d}{2})}\cdot\frac{1}{2}\int_0^\infty d(\ell^2)\frac{(\ell^2)^{\frac{d}{2}-1}}{\left[\ell^2+\Lambda^2\right]^2} =$$

$\Big|\ \xi = \frac{\Lambda^2}{\ell^2+\Lambda^2}$

$\ \ d\xi = -\frac{\Lambda^2}{(\ell^2+m^2)^2}d\ell^2$

$\ \ \ell^2 = \frac{\Lambda^2}{\xi}-\Lambda^2$

$$= \frac{1}{2^{d-1}\pi^{d/2}\Gamma(\frac{d}{2})}\cdot\frac{1}{2}\cdot\int_0^1\frac{d\xi}{\Lambda^2}\cdot\left(\Lambda^2\,\frac{1-\xi}{\xi}\right)^{\frac{d}{2}-1} =$$

$$= \frac{(\Lambda^2)^{\frac{d}{2}-2}}{2^d\pi^{d/2}\Gamma(d/2)}\int_0^1 d\xi\cdot\xi^{1-\frac{d}{2}}(1-\xi)^{\frac{d}{2}-1}$$

can show

Beta-function: $B(\alpha,\beta) = \int_0^1 d\xi\cdot\xi^{\alpha-1}(1-\xi)^{\beta-1} \overset{\text{can show}}{=} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$\Rightarrow \int\frac{d^d\ell_E}{(2\pi)^d}\frac{1}{\left[\ell_E^2+\Lambda^2\right]^2} \overset{\alpha=2-\frac{d}{2},\,\beta=\frac{d}{2}}{=} \frac{\Lambda^{d-4}}{2^d\pi^{d/2}\cancel{\Gamma(\frac{d}{2})}}\cdot\frac{\Gamma(2-\frac{d}{2})\cancel{\Gamma(\frac{d}{2})}}{\underbrace{\Gamma(2)}_{=1}}$$

We finally have..

$$\boxed{\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{[l_E^2+\Lambda^2]^2} = \frac{(\Lambda^2)^{\frac{d}{2}-2}}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right)}.$$

Note that at $d=4$ the integral is indeed divergent and rhs is ∞ too. ($\Gamma(0)=\infty$).

(ii) $$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{[l_E^2+\Lambda^2]^2} = \int \frac{d\, l_E}{(2\pi)^d} \frac{l_E^{d+1}}{l_E^2+\Lambda^2} \int d\Omega_d = \text{as above}$$

$$= \frac{1}{2^d \pi^{d/2} \Gamma(d/2)} \int_0^\infty dl^2 \frac{(l^2)^{d/2}}{[l^2+\Lambda^2]^2} = \frac{(\Lambda^2)^{\frac{d}{2}-1}}{2^d \pi^{d/2}\Gamma(d/2)} \int_0^1 d\xi \cdot \xi^{-d/2}.$$

$$\cdot(1-\xi)^{d/2} = \frac{(\Lambda^2)^{\frac{d}{2}-1}}{(4\pi)^{d/2}\Gamma(\frac{d}{2})} \cdot \frac{\Gamma(1-\frac{d}{2})\,\underbrace{\Gamma(1+\frac{d}{2})}_{=\frac{d}{2}\Gamma(\frac{d}{2})}}{\underbrace{\Gamma(2)}_{=1}} =$$

$$= \frac{(\Lambda^2)^{\frac{d}{2}-1}\Gamma(1-\frac{d}{2})\cdot\frac{d}{2}}{(4\pi)^{d/2}}.$$

Hence $$\boxed{\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{[l_E^2+\Lambda^2]^2} = \frac{(\Lambda^2)^{\frac{d}{2}-1}}{(4\pi)^{d/2}} \cdot \frac{d}{2} \cdot \Gamma\left(1-\frac{d}{2}\right)}!$$

In general γ^μ-matrices & their traces are often affected in d-dimensions (e.g. $\gamma^\mu\gamma^\nu\gamma_\mu = -(d-2)\gamma^\nu$), but we are lucky here as nothing got affected in the above calculation because $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$, $\mathrm{tr}\,\mathbb{1}=4$ are still true in d dimensions.

Using all these results, dimensionally-regularized photon self-energy is:

$$\Pi_2^{\mu\nu}(q) = -4e^2 \int_0 dx \int \frac{d^d l_E}{(2\pi)^d} \frac{-\frac{2}{d} l_E^2 g^{\mu\nu} - 2x(1-x) q^\mu q^\nu + g^{\mu\nu}[l_E^2 + x(1-x)q^2 + m^2]}{[l_E^2 - x(1-x)q^2 + m^2]^2}$$

$$= -4e^2 \int_0^1 dx \int \frac{d^d l_E}{(2\pi)^d} \cdot \left\{ \frac{g^{\mu\nu} l_E^2 \left(1-\frac{2}{d}\right)}{[l_E^2 - x(1-x)q^2 + m^2]^2} + \frac{g^{\mu\nu}[x(1-x)q^2 + m^2] - 2x(1-x) q^\mu q^\nu}{[l_E^2 - x(1-x)q^2 + m^2]^2} \right\} = -4e^2 \int_0^1 dx \cdot$$

$$\left\{ g^{\mu\nu}\left(1-\frac{2}{d}\right) \cdot \frac{(m^2 - x(1-x)q^2)^{\frac{d}{2}-1}}{(4\pi)^{d/2}} \cdot \frac{d}{2} \cdot \Gamma\left(1-\frac{d}{2}\right) + \left[g^{\mu\nu}[x(1-x)q^2 + m^2] - 2x(1-x) q^\mu q^\nu\right] \cdot \frac{(m^2 - x(1-x)q^2)^{\frac{d}{2}-2}}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \right\} = -4e^2 \int_0^1 dx \cdot$$

$$\cdot \frac{(m^2 - x(1-x)q^2)^{\frac{d}{2}-2}}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \cdot \left\{ -g^{\mu\nu}\left(\cancel{m^2} - x(1-x)q^2\right) + g^{\mu\nu}\left[x(1-x)q^2 \cancel{+m^2}\right] - 2x(1-x) q^\mu q^\nu \right\} = -8e^2 \int_0^1 dx \cdot x \cdot (1-x)$$

$$\cdot \frac{(m^2 - x(1-x)q^2)^{\frac{d}{2}-2}}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \cdot \left\{ g^{\mu\nu} q^2 - q^\mu q^\nu \right\}$$

← note that this structure is reproduced!

$\Rightarrow$ as $\Pi_2^{\mu\nu}(q) = \left[g^{\mu\nu} q^2 - q^\mu q^\nu\right] \Pi_2(q^2)$ $\Rightarrow$

$$\Rightarrow \boxed{\Pi_2(q^2) = -8e^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dx \cdot x \cdot (1-x) \cdot \left[m^2 - x(1-x)q^2\right]^{\frac{d}{2}-2}}$$

Now let us expand around $d=4$. Define $\boxed{\varepsilon = 4-d}$.

$$\Rightarrow \Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + o(\varepsilon)$$

$\gamma = 0.5772\ldots$ Euler constant.

$$\Pi_2(q^2) = -8e^2 \frac{1}{(4\pi)^{2-\frac{\varepsilon}{2}}} \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx \cdot x \cdot (1-x) \cdot \left[m^2 - x(1-x)q^2\right]^{-\frac{\varepsilon}{2}}$$

$$= \frac{-8e^2}{(4\pi)^2} \int_0^1 dx \cdot x \cdot (1-x) \cdot \left(1 + \frac{\varepsilon}{2}\ln 4\pi + o(\varepsilon^2)\right)\left(\frac{2}{\varepsilon} - \gamma + o(\varepsilon)\right)\left(1 - \frac{\varepsilon}{2} \cdot \right.$$

$$\left. \cdot \ln\left(m^2 - x(1-x)q^2\right) + o(\varepsilon^2)\right) = -\frac{e^2}{2\pi^2} \cdot \int_0^1 dx \cdot x \cdot (1-x) \cdot \left[\frac{2}{\varepsilon} - \gamma + \right.$$

$$\left. + \ln 4\pi - \ln\left(m^2 - x(1-x)q^2\right) + o(\varepsilon)\right].$$

divergent part.

Hence

$$\boxed{\Pi_2(q^2) = -\frac{2\alpha}{\pi} \int_0^1 dx \cdot x \cdot (1-x) \cdot \left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln\left[m^2 - x(1-x)q^2\right]\right]}.$$

We have now isolated the divergence into $\frac{2}{\varepsilon}$-term.

Let us study some properties of this result.

$$Z_3 = \frac{1}{1-\Pi_2(0)} \Rightarrow \boxed{\delta Z_3 = Z_3 - 1 = -\frac{\alpha}{3\pi}\left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln m^2\right]}.$$