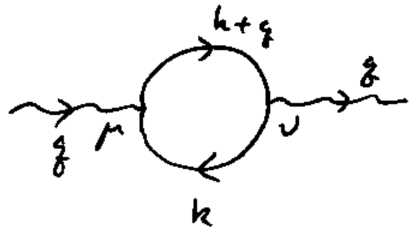
Last time | Calculated vacuum polarization diagram

$$i\Pi_2^{\mu\nu}(q) =$$ 

Used dimensional regularization: $\boxed{\dfrac{d^4 \ell_E}{(2\pi)^4} \to \dfrac{d^d \ell_E}{(2\pi)^d}}$.

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad , \quad \ell^\mu \ell^\nu \to \frac{1}{d}\ell^2 g^{\mu\nu} \to -\frac{1}{d}\ell_E^2 g^{\mu\nu}.$$

The integral converges for some d $\Rightarrow$ integrate, get the answer as a function of d & analytically continue to $d=4$ $\Rightarrow$ get an explicit pole in $\varepsilon = 4-d$:

$$\Pi_2^{\mu\nu}(q) = \left[q^2 g^{\mu\nu} - q^\mu q^\nu \right] \Pi_2(q^2)$$

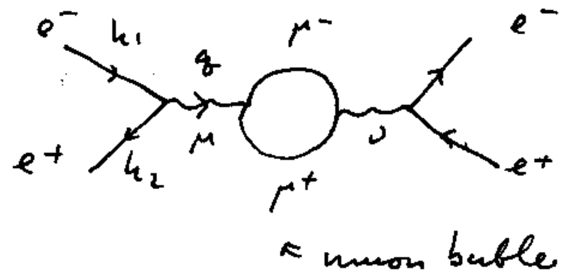
with $$\Pi_2(q^2) = -\frac{2\alpha}{\pi} \int_0^1 dx \cdot x \cdot (1-x) \cdot \left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln\left[m^2 - x(1-x)q^2\right]\right]$$

Leading pole of resummed propagator is at $q^2 = 0$ $\Rightarrow$

$$Z_3 \frac{-i g_{\mu\nu}}{q^2} \simeq \frac{-i g_{\mu\nu}}{q^2[1-\Pi(0)]} \Rightarrow Z_3 = \frac{1}{1-\Pi(0)} \Rightarrow$$

$$\boxed{\delta Z_3 = Z_3 - 1 = -\frac{\alpha}{3\pi}\left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln m^2\right]}$$ field strength renormalization.

<u>Property 1</u> Consider $e^+e^- \to e^+e^-$ forward scattering amplitude at $o(e^2)$:



Optical theorem states that:

$$\sigma_{tot}^{e^+e^-\to\mu^+\mu^-} = \frac{1}{2\mathcal{E}_{k_1}\, 2\mathcal{E}_{k_2}\, |\vec{v}_1 - \vec{v}_2|} \cdot 2 \cdot \operatorname{Im} M(k_1, k_2 \to k_1, k_2)$$

as only the cut  is non-zero.

(as $q^2 > 0 \Rightarrow$  $= 0$.)

In CMS frame we had earlier: $(m_e \approx 0)$

$$\boxed{\sigma_{tot}^{CMS} = \frac{4\pi\, \alpha_{EM}^2}{3s} \sqrt{1 - \frac{4m_\mu^2}{s}} \cdot \left[1 + 2\frac{m_\mu^2}{s}\right]}.$$

Optical th'm gives $\sigma_{tot}^{CMS} = \frac{1}{s} \operatorname{Im} M$

(as $s = 4\mathcal{E}_k^2$)

$$\operatorname{Im} M = (-ie)^2 \cdot \left(\frac{-i}{q^2}\right)^2 \cdot \operatorname{Tr}[\not{k}_1 \gamma_\nu \not{k}_2 \gamma_\mu] \cdot \underbrace{\operatorname{Im} \Pi_2^{\mu\nu}(q)}_{g^{\mu\nu} q^2 \operatorname{Im}\Pi_2(q)} \cdot \frac{1}{4} =$$

↑ propagators

average over incoming quark's polarizations

$$= \frac{e^2}{4q^2} \underbrace{\operatorname{Tr}[\not{k}_1 \gamma^\nu \not{k}_2 \gamma_\nu]}_{-2\cdot 4 \cdot k_1 \cdot k_2} \operatorname{Im}\Pi_2(q) = \frac{e^2}{4s}(-8) \cdot \overbrace{k_1 \cdot k_2}^{s/2} \operatorname{Im}\Pi_2(q)$$

$$\Rightarrow \operatorname{Im} M = -e^2 \operatorname{Im} \Pi_2(q^2)$$

$$\Rightarrow \sigma_{tot}^{CMS} = -\frac{e^2}{s} \operatorname{Im} \Pi_2(q^2) = -\frac{e^2}{s} \cdot \frac{2\alpha}{\pi} \cdot \int_0^1 dx \cdot x \cdot (1-x) \cdot$$

$$\cdot \operatorname{Im} \left\{ \ln \left[m_\mu^2 - x(1-x) q^2 \right] \right\}.$$

One can see that Im part exists only if at some x

$m_\mu^2 < x(1-x) q^2 \Rightarrow m_\mu^2 < x(1-x) s \leqslant \frac{s}{4} \Rightarrow s \geqslant 4 m_\mu^2$

~ usual threshold for pair production!

To find Im part had to keep $i\varepsilon \Rightarrow$ if done right

have $q^2 \to q^2 + i\varepsilon \Rightarrow \operatorname{Im}\left\{ \ln \left[m_\mu^2 - x(1-x) q^2 \right] \right\} \longrightarrow$

$$\to \operatorname{Im} \left\{ \ln \left[m_\mu^2 - x(1-x)(q^2 + i\varepsilon) \right] \right\} = -\pi \, \theta\left(x(1-x) q^2 - m_\mu^2 \right)$$

$$\Rightarrow \sigma_{tot}^{CMS} = \frac{e^2}{s} \cdot \frac{2\alpha}{\cancel{\pi}} \cdot \cancel{\pi} \cdot \int_0^1 dx \cdot x \cdot (1-x) \cdot \theta\left(x(1-x) s - m_\mu^2 \right)$$

Define $y = x - \frac{1}{2} \Rightarrow x(1-x) = \left(y + \frac{1}{2}\right)\left(\frac{1}{2} - y\right) = \frac{1}{4} - y^2$

$\Rightarrow x(1-x) s - m_\mu^2 \geqslant 0$ becomes $\left(\frac{1}{4} - y^2\right) s \geqslant m_\mu^2 \Rightarrow$

$$\Rightarrow \frac{1}{4} - y^2 \geqslant \frac{m_\mu^2}{s} \Rightarrow |y| \leqslant \frac{1}{2} \sqrt{1 - \frac{4 m_\mu^2}{s}} \Rightarrow$$

$$\sigma_{tot}^{CMS} = \frac{2 e^2 \alpha}{s} \cdot \int_{-\frac{1}{2}\sqrt{1 - \frac{4 m_\mu^2}{s}}}^{\frac{1}{2}\sqrt{1 - \frac{4 m_\mu^2}{s}}} dy \cdot \left(\frac{1}{4} - y^2 \right) = \frac{2 \cdot 4\pi \cdot \alpha_{EM}^2}{s} \cdot \left[\frac{1}{4} \sqrt{1 - \frac{4 m_\mu^2}{s}} \right.$$

$$- \frac{1}{3}\cdot 2\cdot\frac{1}{2^3}\left(1-\frac{4m_\mu^2}{s}\right)^{3/2}\Bigg] = \frac{8\pi\alpha_{EM}^2}{s}\sqrt{1-\frac{4m_\mu^2}{s}}\;\cdot$$

$$\cdot\left[\frac{1}{4} - \frac{1}{4\cdot 3}\left(1-\frac{4m_\mu^2}{s}\right)\right] = \frac{2\pi\alpha_{EM}^2}{s}\sqrt{1-\frac{4m_\mu^2}{s}}\cdot\Bigg[1 -$$

$$-\frac{1}{3} + \frac{4}{3}\frac{m_\mu^2}{s}\Bigg] = \frac{4\pi\alpha_{EM}^2}{3s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[1+2\frac{m_\mu^2}{s}\right].$$

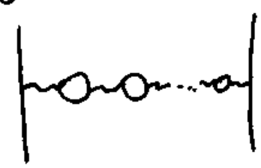
exactly as desired!

<u>Property 2</u> Consider Coulomb potential between electron and positron:

$$V(r) = \int\frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}}\;\frac{-e_0^2}{|\vec{q}|^2}.$$

$\uparrow$ bare propagator.

Including 1PI graphs gives



$$V(r) = \int\frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}}\;\frac{-e^2}{|\vec{q}|^2\left(1-[\Pi(q^2)-\Pi(0)]\right)}$$

$$\Pi_2(q^2) - \Pi_2(0) = +\frac{2\alpha}{\pi}\int_0^1 dx\cdot x\cdot(1-x)\cdot\ln\left(\frac{m^2+x(1-x)|\vec{q}|^2}{m^2}\right)\simeq$$

$$\simeq\left(\text{if } |\vec{q}|\ll m\right)\simeq\frac{2\alpha}{\pi}\int_0^1 dx\cdot x\cdot(1-x)\cdot\frac{x(1-x)|\vec{q}|^2}{m^2} = \frac{2\alpha}{\pi m^2}|\vec{q}|^2\cdot$$

$$\cdot\int_0^1 dx\cdot x^2\cdot(1-x)^2 = \frac{2\alpha}{\pi m^2}|\vec{q}|^2\cdot\Big[\underbrace{\frac{1}{3}-\frac{2}{4}}_{-\frac{1}{6}}+\frac{1}{5}\Big] = \frac{2\alpha}{30\pi m^2}|\vec{q}|^2$$

$$\Rightarrow V(r)\Big|_{r\gg\frac{1}{m}} \simeq \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}}\, \frac{-e^2}{|\vec{q}|^2}\cdot\left[1+\frac{\alpha}{15\pi}\frac{|\vec{q}|^2}{m^2}\right]$$

$$= \underbrace{-\frac{\alpha}{r}}_{\text{usual Coulomb potential}} - \frac{4}{15}\frac{\alpha^2}{m^2}\underbrace{\int\frac{d^3q}{(2\pi)^3}\,e^{i\vec{q}\cdot\vec{r}}}_{\delta^3(\vec{r})}$$

$$\Rightarrow \boxed{V(r)\Big|_{r\gg\frac{1}{m}} \simeq -\frac{\alpha}{r} \underbrace{- \frac{4}{15}\frac{\alpha^2}{m^2}\delta^3(\vec{r})}_{\text{Uehling, 1930's. contributes to Lamb shift.}}}$$

Now, let us calculate the running of the coupling:

$$\alpha(q^2) = \frac{\alpha}{1-[\Pi(q^2)-\Pi(0)]}$$

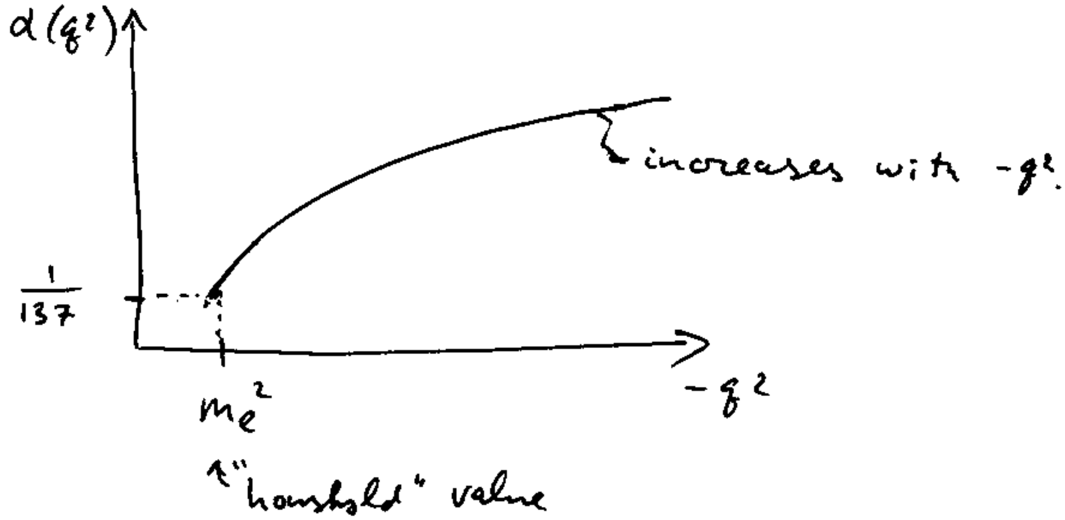
$$\Pi_2(q^2)-\Pi_2(0) = \frac{2\alpha}{\pi}\int_0^1 dx\cdot x\cdot(1-x)\cdot\ln\left(\frac{m^2-x(1-x)q^2}{m^2}\right)\simeq$$

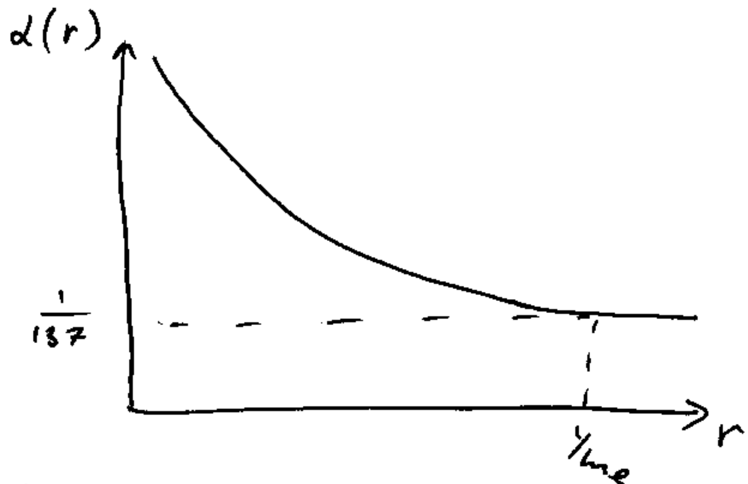
$$\simeq\left(-q^2\gg m^2\right)\simeq\frac{2\alpha}{\pi}\int_0^1 dx\cdot x\cdot(1-x)\cdot\ln\left(\frac{-q^2\,x(1-x)}{m^2}\right)$$

$$= \frac{2\alpha}{\pi}\cdot\Big[\frac{1}{6}\ln\left(\frac{-q^2}{m^2}\right)+\underbrace{\int_0^1 dx\cdot x\cdot(1-x)\ln[x(1-x)]}_{-5/18}\Big]$$

$$= \frac{\alpha}{3\pi}\left[\ln\left(\frac{-q^2}{m^2}\right)-\frac{5}{3}\right]$$

$$\Rightarrow \quad \alpha(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln\left(\frac{-q^2}{m^2 e^{5/3}}\right)}$$





In coordinate space:

Electron-positron pairs pop out of the vacuum to screen the effective charge of the electron, just like in a dielectric: