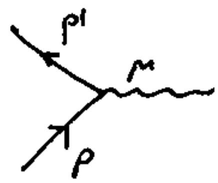
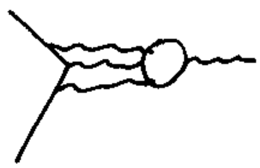
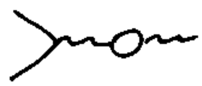
Vertex Correction in QED & Ward-Takahashi identity.

We have resummed 1-loop corrections to electron and photon propagators. What about electron-photon vertex?  $-ie\gamma^\mu$

As usual start by defining the sum of all 1PI diagrams:

Def. $-ie\Gamma^\mu(p',p) =$  $+$  $+$ 

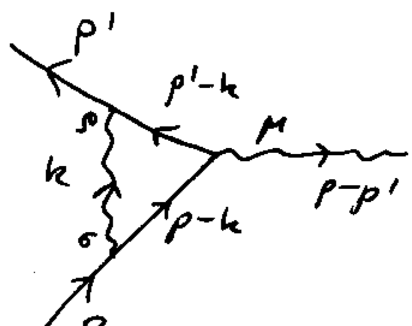
$+ \ldots +$  $+ \ldots$.

Drop reducible graphs like  or 

$\Rightarrow$ go into propagator corrections.

Write
$$\Gamma^\mu(p',p) = \underbrace{\gamma^\mu}_{\text{leading-order}} + \underbrace{\Lambda^\mu(p',p)}_{\text{all h.o. corrections}}$$

Start at one-loop:



$$\Lambda_2^\mu = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\rho\sigma}}{k^2+i\varepsilon} \cdot \gamma^\rho \frac{i}{\not p' - \not k - m} \cdot \gamma^\mu \frac{i}{\not p - \not k - m} \gamma^\sigma$$

$$\boxed{\Lambda^\mu_2(p',p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\varepsilon} \gamma^\rho \frac{1}{\not p' - \not k - m} \gamma^\mu \frac{1}{\not p - \not k - m} \gamma_\rho}$$

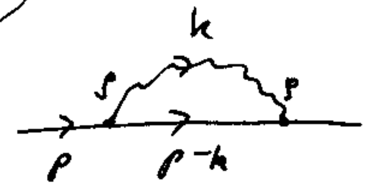
$\sim$ at most $\int^\Lambda \frac{d^4k}{k^4} \sim \ln\Lambda \sim$ logarithmic divergence.

(i) First consider the case when $p' = p$:

$$\Lambda^\mu_2(p,p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\varepsilon} \gamma^\rho \frac{1}{\not p - \not k - m} \gamma^\mu \frac{1}{\not p - \not k - m} \gamma_\rho .$$

Compare this with electron's self-energy:

$$\boxed{\Sigma_2(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\varepsilon} \gamma^\rho \frac{1}{\not p - \not k - m} \gamma_\rho}$$



Differentiate $\Sigma(p)$:

$$\frac{\partial \Sigma_2(p)}{\partial p^\mu} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\varepsilon} \gamma^\rho \frac{\partial}{\partial p^\mu}\left(\frac{1}{\not p - \not k - m}\right) \gamma_\rho$$

$$\frac{\partial}{\partial p^\mu}\left(\frac{1}{\not p - \not k - m}\right) = -\frac{1}{\not p - \not k - m}\left[\frac{\partial}{\partial p^\mu}(\not p - \not k - m)\right]\frac{1}{\not p - \not k - m} = -\frac{1}{\not p - \not k - m}\gamma_\mu \frac{1}{\not p - \not k - m}.$$

(as $\partial(S^{-1}\cdot S) = 0 = (\partial S^{-1})S + S^{-1}(\partial S) \Rightarrow (\partial S^{-1}) = -S^{-1}(\partial S)S^{-1}$.)

$$\Rightarrow -\frac{\partial \Sigma_2(p)}{\partial p^\mu} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\varepsilon} \gamma^\rho \frac{1}{\not p - \not k - m} \gamma_\mu \frac{1}{\not p - \not k - m} \gamma_\rho = \Lambda^2_\mu(p,p).$$

$$\Rightarrow \boxed{-\frac{\partial \Sigma(p)}{\partial p^\mu} = \Lambda_\mu(p,p)}.$$ Ward identity (true to all orders in e)

Equivalently, $\Gamma_\mu(p,p) = \frac{\partial}{\partial p^\mu}[\not p - m - \Sigma(p)]$ relation between vertex & inverse propagator.

(ii) What about the case when $p' \neq p$?

Define $q^\mu = p^\mu - p'^\mu$ and consider:

$$q^\mu \Lambda^2_\mu(p',p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\varepsilon} \gamma^\rho \frac{1}{\not p' - \not k - m} \underbrace{(\not p - \not p')}_{(\not p - \not k - m) - (\not p' - \not k - m)} \frac{1}{\not p - \not k - m} \gamma_\rho$$

$$= -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\varepsilon} \left\{ \gamma^\rho \frac{1}{\not p' - \not k - m} \gamma_\rho - \gamma^\rho \frac{1}{\not p - \not k - m} \gamma_\rho \right\} =$$

$$= \Sigma_2(p') - \Sigma_2(p)$$

$$\Rightarrow \boxed{q^\mu \Lambda_\mu(p',p) = \Sigma(p') - \Sigma(p)}.$$ Ward-Takahashi identity

Going to $\Gamma_\mu(p',p)$ have

$$q^\mu \Gamma_\mu(p',p) = q^\mu \gamma_\mu + q^\mu \Lambda_\mu(p',p) = \not q + \Sigma(p') - \Sigma(p)$$

$$= \left[\not p - m - \Sigma(p)\right] - \left[\not p' - m - \Sigma(p')\right] = i\left[S^{-1}(p) - S^{-1}(p')\right]$$

where $S(p) = \dfrac{i}{\not p - m - \Sigma(p)}$ is the dressed electron's propagator.

$$\Rightarrow \boxed{-i q^\mu \Gamma_\mu(p',p) = S^{-1}(p) - S^{-1}(p')}$$ also Ward-Takahashi identity.

$\Rightarrow$ True to all orders (see e.g. Sterman's book)

Def. Vertex renormalization factor Z_1:

$$\lim_{q\to 0} \Gamma^\mu(p-q,p) = \frac{1}{Z_1}\gamma^\mu.$$

Ward-Takahashi identity:

$$-iq^\mu \Gamma_\mu(p-q,p) = S^{-1}(p) - S^{-1}(p-q)$$

in the $q\to 0$ limit l.h.s. $= \frac{-i}{Z_1}\not{q}$.

To evaluate rhs remember that $S(p) \simeq Z_2 \frac{i}{\not{p}-m_{phys}}$
for $p^2 \simeq m_{phys}^2$

$$\Rightarrow \quad S^{-1}(p) = \frac{-i}{Z_2}(\not{p}-m_{phys}) \quad \Rightarrow$$

$$S^{-1}(p) - S^{-1}(p-q) = \frac{-i}{Z_2}\left[\cancel{\not{p}} - \cancel{m_{phys}} - \cancel{\not{p}} + \not{q} + \cancel{m_{phys}}\right]$$

$$= -\frac{i}{Z_2}\not{q} \quad \Rightarrow \quad -\frac{i}{Z_1}\not{q} = -\frac{i}{Z_2}\not{q} \quad \Rightarrow$$

$$\boxed{Z_1 = Z_2}$$

$\Rightarrow$ Ward-Takahashi identity guarantees that the vertex renormalization factor in QED (Z_1) is the same as the electron's renormalization factor Z_2!

$\Rightarrow$ Can be confirmed by an explicit calculation.

Renormalization of QED.

Start with the QED Lagrangian written in terms of bare fields, mass & coupling:

$$\mathcal{L} = \bar{\psi}_0 [i \not\partial - m_0] \psi_0 - \frac{1}{4} F^0_{\mu\nu} F^{0\mu\nu} - e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A^0_\mu$$

ψ_0 ~ bare electron field

A^0_μ ~ bare photon field, $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$.

e_0 ~ bare coupling, m_0 ~ bare electron's mass

Rescale: $\psi_0 = \sqrt{Z_2}\, \psi$, $A^0_\mu = \sqrt{Z_3}\, A_\mu$ with

ψ, A_μ ~ renormalized fields.

The Lagrangian becomes:

$$\mathcal{L} = -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + Z_2 \bar{\psi} [i\not\partial - m_0] \psi - e_0 Z_2 Z_3^{1/2} \bar{\psi} \gamma^\mu \psi A_\mu.$$

We want the interaction vertex to be $\Gamma_\mu (q=0) = \frac{1}{Z_1} \gamma_\mu$

$\Rightarrow$ define Z_1 by

$$\boxed{e_0 Z_2 Z_3^{1/2} = e Z_1}$$

e ~ renormalized (physical) charge.

Note that due to Ward identity $Z_1 = Z_2 \Rightarrow \boxed{e = e_0 \sqrt{Z_3}}$.
as we defined before.

We have

$$\mathcal{L} = -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + Z_2 \bar{\psi} [i\partial\!\!\!/ - m_0]\psi - e Z_1 \bar{\psi}\gamma^\mu \psi A_\mu .$$

We want to separate the Lagrangian without Z's from the Lagrangian with Z's:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} [i\partial\!\!\!/ - m_0]\psi - e \bar{\psi}\gamma^\mu \psi A_\mu$$

$$- \frac{1}{4}(Z_3 - 1) F_{\mu\nu} F^{\mu\nu} + (Z_2 - 1)\bar{\psi} [i\partial\!\!\!/ - m_0]\psi - (Z_1 - 1) e \bar{\psi}\gamma^\mu \psi A_\mu .$$

We really want to have the physical mass m in the first part:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} [i\partial\!\!\!/ - m]\psi - e \bar{\psi}\gamma^\mu \psi A_\mu$$

$$- \frac{1}{4}(Z_3 - 1) F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left[(Z_2 - 1) i\partial\!\!\!/ - Z_2 m_0 + m\right]\psi - (Z_1 - 1) e \bar{\psi}\gamma^\mu \psi A_\mu$$

Define:

$$\boxed{\delta_3 \equiv Z_3 - 1 , \quad \delta_2 \equiv Z_2 - 1, \quad \delta_m \equiv Z_2 m_0 - m, \quad \delta_1 \equiv Z_1 - 1 = \frac{e_0}{e} Z_2 Z_3^{1/2} - 1}$$

Such that

$$\boxed{\begin{aligned}\mathcal{L} = &-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} [i\partial\!\!\!/ - m]\psi - e \bar{\psi}\gamma^\mu \psi A_\mu \\ &- \frac{1}{4}\delta_3 F_{\mu\nu} F^{\mu\nu} + \bar{\psi}[i\delta_2 \partial\!\!\!/ - \delta_m]\psi - e\delta_1 \bar{\psi}\gamma^\mu\psi A_\mu\end{aligned}}$$

QED Lagrangian written in terms of renormalized fields and physical mass & charge.

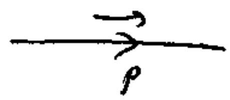
first line ~ renormalized Lagrangian

second line ~ "counterterms".

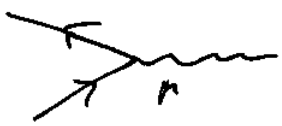
(Sometimes people say that they "added" counterterms
⇒ not true, it is the same QED Lagrangian split
in two.)

Feynman rules for the renormalized Lagrangian are:

$$\frac{-i g_{\mu\nu}}{q^2+i\varepsilon}$$



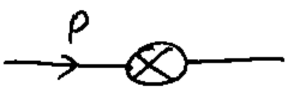
$$= \frac{i}{\not p - m}$$



$$-ie\gamma^\mu$$

same as before

$$-i\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]\delta_3$$



$$i\left(\not p\, \delta_2 - \delta_m\right)$$

$$-ie\,\delta_1 \gamma^\mu$$

new vertices (counterterms)

Counterterms cancel the infinities of the perturbation theory. They may also contain finite parts, adjusted to satisfy renormalization conditions.