Last time | Vertex Corrections (in QED) and Ward-Takahashi identity

Def. $-ie\,\Gamma^\mu(p',p) =$  $\Rightarrow \Gamma^\mu_{(p',p)} = \gamma^\mu + \Lambda^\mu(p',p)$.

Showed that $\boxed{-\dfrac{\partial\Sigma(p)}{\partial p^\mu} = \Lambda_\mu(p,p)}$ Ward identity

or, equivalently, $\boxed{-i\Gamma_\mu(p,p) = \dfrac{\partial}{\partial p^\mu} S'^{-1}(p)}$, $S'(p) = \dfrac{i}{\not p - m - \Sigma(p)}$.

More generally $\boxed{q^\mu \Lambda_\mu(p',p) = \Sigma(p') - \Sigma(p)}$ Ward-Takahashi identity

or $\boxed{-iq^\mu\,\Gamma_\mu(p',p) = S'^{-1}(p) - S'^{-1}(p')}$, $q^\mu = p^\mu - p'^\mu$. relates dressed vertex to dressed propagators.

Def. $\lim\limits_{q\to 0} \Gamma^\mu(p-q,p) = \dfrac{1}{Z_1}\gamma^\mu$ $\Rightarrow$ using Ward-Takahashi identity we showed that $\boxed{Z_1 = Z_2}$.

Renormalization of QED (cont'd)

$$\mathcal{L}_{QED} = \bar\psi_0[i\not\partial - m_0]\psi_0 - \frac{1}{4}F^0_{\mu\nu}F^{0\mu\nu} - e_0\bar\psi_0\gamma^\mu\psi_0 A^0_\mu$$

all fields and parameters are "bare".

Def. Dressed fields $\psi_0 = \sqrt{Z_2}\,\psi$, $A^0_\mu = \sqrt{Z_3}\,A_\mu$,

physical coupling $e Z_1 = e_0 Z_2 Z_3^{1/2}$.

physical mass m.

QED Lagrangian becomes:

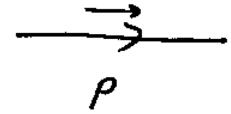
$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i\not{\partial} - m]\psi - e\bar{\psi}\gamma^\mu\psi A_\mu$$
$$-\frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i\delta_2\not{\partial} - \delta_m]\psi - e\delta_1\bar{\psi}\gamma^\mu\psi A_\mu$$

where
$$\delta_3 = Z_3 - 1,\quad \delta_2 = Z_2 - 1,\quad \delta_m = Z_2 m_0 - m,$$
$$\delta_1 = Z_1 - 1 = \frac{e_0}{e} Z_2 Z_3^{1/2} - 1$$
~ counterterms.

Feynman rules:

"old" rules

$\dfrac{-i g_{\mu\nu}}{q^2 + i\varepsilon}$,

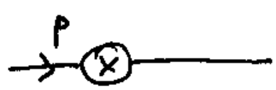


$= \dfrac{i}{\not{p} - m}$

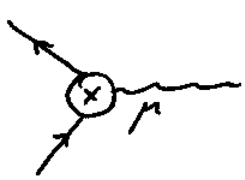
$-ie\gamma^\mu$

new vertices (counterterms)

$-i\delta_3[q^2 g^{\mu\nu} - q^\mu q^\nu]$



$i(\not{p}\delta_2 - \delta_m)$



$-ie\delta_1\gamma^\mu$

$\Rightarrow$ Field theory can not predict particle masses or the coupling constant (it can predict momentum dependence of the coupling): these are external parameters. We can adjust bare parameters / counterterms to make this work.

QED renormalization conditions: ("on-shell scheme")

(i) We want that after renormalization the electron propagator is  $= \dfrac{i}{\not p - m} + \begin{pmatrix}\text{terms regular} \\ \text{at } p^2 = m^2\end{pmatrix}$.

$$\Rightarrow \boxed{\begin{array}{l} \Sigma(p)\Big|_{p^2=m^2} = 0 \\[2ex] \dfrac{\partial \Sigma(p)}{\partial \not p}\Bigg|_{\not p = m} = 0 \end{array}}$$

$\sim$ pole at $p^2 = m^2$ $\left(\text{as } S(p) = \dfrac{i}{\not p - m - \Sigma(p)}\right)$.

$\sim$ residue $= i$ at $\not p = m$ pole.

(Remember, before renormalization we had $\dfrac{1}{Z_2} = 1 - \dfrac{\partial \Sigma}{\partial \not p}\Big|_{\not p = m_{phys}}$)

(ii) We want the photon propagator to be

 $= \dfrac{-i g_{\mu\nu}}{q^2 + i\varepsilon}$

$\Rightarrow \boxed{\Pi(q^2 = 0) = 0}$ $\sim$ residue $= 1$ at $q^2 = 0$ pole.

(iii) We want electron charge to be $= e$.

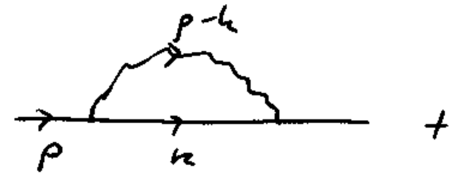
 $= -ie\gamma^\mu \Rightarrow \boxed{\Gamma^\mu(q=0) = \gamma^\mu}$

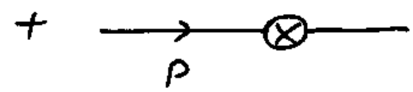
$\Rightarrow$ conditions (i) & (ii) fix $\delta_2, \delta_m, \delta_3$ with (iii) fixing δ_1.

One-loop Structure of QED.

Start with electron self-energy:

$$\Sigma_2(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2+i\varepsilon} \cdot \gamma^\rho \cdot \frac{1}{\not k - m} \gamma_\rho .$$





$$\Rightarrow -i\Sigma(p) = -i\Sigma_2(p) + i(\not p \delta_2 - \delta_m)$$

$$\Rightarrow \Sigma(p) = \Sigma_2(p) - \not p \delta_2 + \delta_m$$

Calculate $\Sigma_2(p)$ using dim. reg.

$$\Sigma_2(p) = -ie^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p-k)^2+i\varepsilon} \frac{\gamma^\mu(\not k + m)\gamma_\mu}{k^2-m^2+i\varepsilon} .$$

In d-dimensions $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ still $\Rightarrow \gamma^\mu\gamma_\mu = \delta^\mu_\mu = d$

$$\gamma^\mu\gamma^\nu\gamma_\mu = \{\gamma^\mu, \gamma^\nu\}\gamma_\mu - \gamma^\nu\gamma^\mu\gamma_\mu = 2\gamma^\nu - d\gamma^\nu = $$

$$= (2-d)\gamma^\nu$$

$$\Rightarrow \begin{aligned} \gamma_\mu\gamma^\mu &= d \\ \gamma^\mu\gamma^\nu\gamma_\mu &= (2-d)\gamma^\nu \end{aligned}$$

$$\Rightarrow \Sigma_2(p) = -ie^2 \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)\not k + dm}{[(p-k)^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]}.$$

$\Rightarrow$ introduce Feynman parameters & do Wick rotation & integrate over momenta to get (cf. Peskin (10.41)):

$$\Sigma_2(p) = \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2 - d/2)[(2-d)x\not p + dm]}{[(1-x)m^2 - x(1-x)p^2]^{2-d/2}}.$$

$$\Rightarrow \Sigma(p)\Big|_{\substack{p^2 = m^2 \\ \not p = m}} = 0 = \Sigma_2(p)\Big|_{\substack{p^2 = m^2 \\ \not p = m}} - m\delta_2 + \delta_m$$

$$\Rightarrow m\delta_2 - \delta_m = \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2-\frac{d}{2})[2x + d(1-x)]m}{[(1-x)^2 m^2]^{2-d/2}}.$$

$$\frac{\partial \Sigma}{\partial \not p}\Big|_{\not p = m} = \frac{\partial \Sigma_2}{\partial \not p}\Big|_{\not p = m} - \delta_2 = 0$$

$$\Rightarrow \delta_2 = \frac{\partial \Sigma_2}{\partial \not p}\Big|_{\not p = m} = \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2 - d/2)}{[(1-x)^2 m^2]^{2-d/2}} \cdot \Big\{(2-d)x + \frac{(2-\frac{d}{2})[2xm + (1-x)dm]}{[(1-x)^2 m^2]} x(1-x)2m\Big\}$$

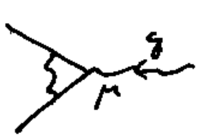
$\Rightarrow$ δ_2 and δ_m are fixed.

Condition (ii) gives:  +  =

$$\Big[i \underbrace{\Pi_2^{\mu\nu}(q)}_{[q^2 g^{\mu\nu} - q^\mu q^\nu]\Pi_2(q^2)} - i\,[q^2 g^{\mu\nu} - q^\mu q^\nu]\,\delta_3 \Big]\Big|_{q^2=0} = 0$$

$$\Rightarrow \Pi(q^2) = \Pi_2(q^2) - \delta_3 \Rightarrow \Pi(q^2=0) = 0 \text{ gives}$$

$$\delta_3^{\text{on-shell}} = \Pi_2(q^2=0) = -\frac{\alpha_{EM}}{3\pi}\Big[\frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln m^2\Big].$$

Condition (iii) yields:  +  =

$$= -ie\,\Gamma^\mu(q) = -ie\,\big[\Gamma_2^\mu(q) + \delta_1 \gamma^\mu\big]. \Rightarrow \text{want}$$

$$\underbrace{\Gamma_2^\mu(q=0)}_{\text{"}\frac{1}{Z_1}\gamma^\mu} + \delta_1 \gamma^\mu = \gamma^\mu$$

$$\text{"}\frac{1}{Z_1}\gamma^\mu \underset{\substack{\uparrow \\ \text{Ward}}}{=} \frac{1}{Z_2}\gamma^\mu \Rightarrow \delta_1 = 1 - \frac{1}{Z_2} \approx Z_2 - 1 = \delta Z_2 = \delta_2$$

$\Rightarrow$ $\delta_1 = \delta_2$ as expected from Ward identity.

$\Rightarrow$ fixed all counterterms: theory is renormalized at one loop.

$\Rightarrow$ one can show that there is no other one-loop divergences in QED:

 $= 0$,  included ,  $= 0$ (Furry's th'm) ,

$\sim \int^{\Lambda} \frac{d^4k}{k^4} \sim \ln \Lambda \Rightarrow$ in fact finite (can show).

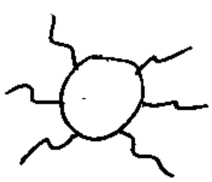
In general can characterize the diagram by its <u>superficial degree of divergence</u>: $D = 4L - P_e - 2P_\gamma$

L = # loops (each loop gives d^4k)

P_e = # of electron propagators (each fermion prop. gives $1/k$)

P_γ = # — photon — (each gives $1/k^2$).

$\Rightarrow$ the diagram should diverge at most as Λ^D.
(if $D<0$ + subdiagrams $\Rightarrow$ convergent diagram Weinberg's th'm)

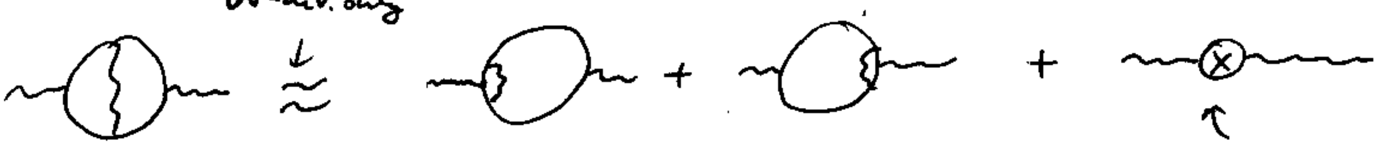


$\sim \Lambda^{4\cdot 1-6} \sim \Lambda^{-2} \sim \frac{1}{\Lambda^2} \Rightarrow$ finite

$L=1,\ P_e=6,\ P_\gamma=0$ (all other multi-leg 1-loops are finite too)

What about multi-loop graphs? One can show that UV divergences are removed by counterterms:

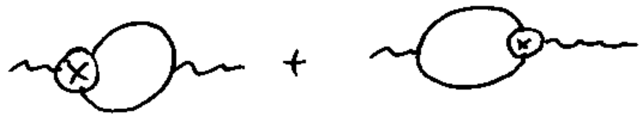
UV-div. only



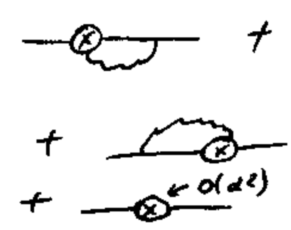
counterterm at $O(\alpha^2)$

1st & 2nd terms are

$\Rightarrow$ removed by



$\Rightarrow$ removed by



$\Rightarrow$ QED is renormalizable!