Brief Review of Regularization and Renormalization

<u>Regularization</u>: a procedure to make divergent integrals finite in order to determine ∞ part and a finite piece.

Two main types of regularization:

- <u>Pauli-Villars</u>: introduce "new" massive particles with masses $M_i^2 = m^2 + \alpha_i \cdot M^2$.
  ↑ bare mass ↖ new large mass parameter

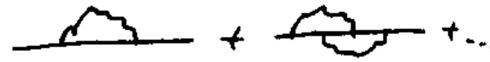
  Each propagator becomes

$$\frac{i}{k^2-m^2+i\varepsilon} \to \sum_i c_i \frac{i}{k^2-M_i^2+i\varepsilon}.$$

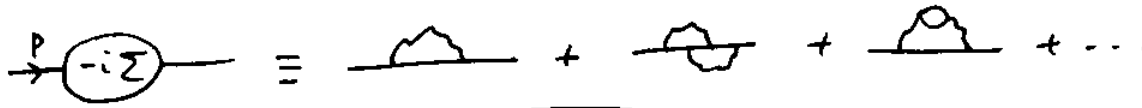
  We then take $M \to \infty$ limit keeping ∞ and finite terms only.

- <u>Dimensional regularization</u>: replace $4 \to d$ in the integrals, traces, ... Get some answer as a function of d & take $d \to 4$ limit ($\varepsilon = 4-d \to 0$) keeping divergent (e.g. $\frac{1}{\varepsilon}, \frac{1}{\varepsilon^2}, \ldots$) and finite (e.g. $\ln 4\pi, \frac{5}{3}, \ldots$) terms.

We worked out the example of <u>QED</u>:

<u>Electron propagator corrections</u>: 

$-i\Sigma(p)$ = sum of all 1PI graphs



got $\boxed{S(p) = \dfrac{i}{\not p - m_0 - \Sigma(p)}}$ ~ dressed propagator.

We want
$$\frac{i}{\not p - \underset{\substack{\uparrow \\ \text{bare mass}}}{m_0} - \Sigma(p)} = Z_2 \frac{i}{\not p - \underset{\substack{\uparrow \\ \text{physical mass}}}{m}} + \left(\text{finite at } p^2 = m^2 \text{ terms}\right).$$

We calculated $-i\Sigma_2(p)$ =  (one-loop)

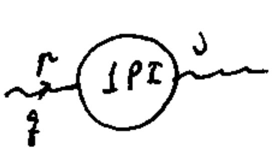
and found:
$$\boxed{\delta m \equiv m - m_0 = \frac{3\alpha_{EM}}{4\pi} m_0 \left\{ \ln\left(\frac{M^2}{m_0^2}\right) + \frac{1}{2} \right\}}$$

$$\boxed{\delta Z_2 = Z_2 - 1 = -\frac{\alpha_{EM}}{4\pi} \left\{ \ln\left(\frac{M^2}{m_0^2}\right) + \frac{9}{2} - 4\int_0^1 \frac{dx}{1-x} \right\}}$$

(we used Pauli-Villars regularization).

$\boxed{\delta Z_2 = -\dfrac{\alpha_{EM}}{2\pi} \cdot \dfrac{1}{\varepsilon} + \text{const}}$ ← if we had used dim. reg.

<u>Photon propagator corrections</u>:

$i\Pi^{\mu\nu}(q)$ =  $\Rightarrow \boxed{\Pi^{\mu\nu}(q) = \left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]\Pi(q^2)}$

such that the dressed photon propagator is

$$\boxed{D_{\mu\nu}(q) = \frac{-i g_{\mu\nu}}{q^2\left[1 - \Pi(q^2)\right]} + \left(q^\mu q^\nu\text{-terms}\right)}$$

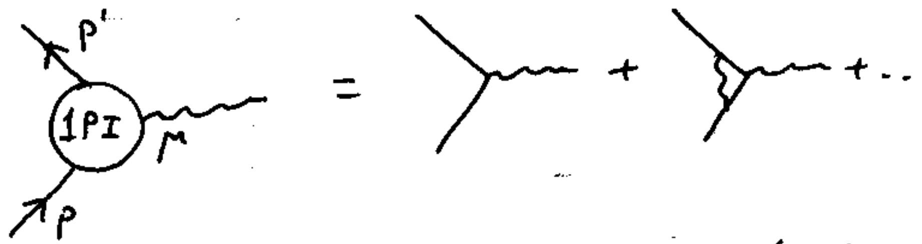
Want $D_{\mu\nu}(q) = Z_3 \frac{-i g_{\mu\nu}}{q^2 + i\varepsilon} \Rightarrow \boxed{Z_3 = \frac{1}{1-\Pi(0)}}$

Used <u>dim. reg.</u> to find $\boxed{\delta Z_3 = Z_3 - 1 = -\frac{\alpha}{3\pi}\left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln m^2\right]}$

$\varepsilon = 4 - d$.

<u>QED Vertex Correction</u>:

$-ie\,\Gamma^\mu(p', p) =$



$q^\mu = p^\mu - p'^\mu$

$\boxed{-i q^\mu \Gamma_\mu(p', p) = S^{-1}(p) - S^{-1}(p')}$ Ward-Takahashi identity

Defining Z_1 by $\lim_{q \to 0} \Gamma^\mu(p-q, p) \equiv \frac{1}{Z_1}\gamma^\mu$ we used Ward-Takahashi identity to prove that $\boxed{Z_1 = Z_2}$ in QED.

<u>Renormalization</u>: rearrangement of perturbation theory in such a way that at each order in the coupling all observables are finite.

For QED: start with bare Lagrangian:

$$\mathcal{L}_{QED} = \bar\psi_0\left[i\not\partial - m_0\right]\psi_0 - \frac{1}{4}F^0_{\mu\nu}F^{0\mu\nu} - e_0\bar\psi_0\gamma^\mu\psi_0 A^0_\mu .$$

Define physical fields: $\psi = \frac{1}{\sqrt{Z_2}}\psi_0$, $A_\mu = \frac{1}{\sqrt{Z_3}}A^0_\mu$

and the physical coupling: $\boxed{e = e_0\frac{Z_2 Z_3^{1/2}}{Z_1} = e_0\sqrt{Z_3}}$

as $Z_1 = Z_2$

One then has

$$\boxed{\begin{aligned}\mathcal{L}_{QED} = & -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar\psi[i\not\partial - m]\psi - e\bar\psi\gamma^\mu\psi A_\mu \\ & -\frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + \bar\psi[i\delta_2\not\partial - \delta_m]\psi - e\,\delta_1\bar\psi\gamma^\mu\psi A_\mu\end{aligned}}$$

last line ~ counterterms.

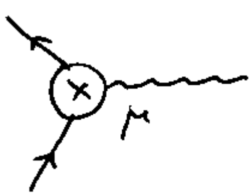
$$\boxed{\delta_3 = Z_3 - 1,\quad \delta_2 = Z_2 - 1,\quad \delta_1 = Z_1 - 1,\quad \delta_m = Z_2 m_0 - m.}$$

New vertices in perturbation theory:

$-ie\gamma^\mu$ ~ old vertex

$-i\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]\delta_3$

$i(\not p\,\delta_2 - \delta_m)$



$-ie\delta_1\gamma^\mu$

} counterterms (new vertices)

How do we choose counterterms?

We want  +  to be finite.

$-i\Sigma_2(p)$

However:

$$-i\Sigma(p) = -i\Sigma_2(p) + i(\not p\,\delta_2 - \delta_m)$$

In dim. reg. $\Sigma_2(p) = -\frac{\alpha_{EM}}{2\pi}\frac{1}{\varepsilon}(\not p - 4m) + \text{finite}$ (B5)

Check:

$$S(p) = \frac{1}{\not p - m - \Sigma(p)} = \frac{1}{\not p - m + \frac{\alpha}{2\pi}\frac{1}{\varepsilon}(\not p - 4m)} = \frac{1}{\not p\left(1+\frac{\alpha}{2\pi}\frac{1}{\varepsilon}\right) - m\left(1+\frac{\alpha}{\pi}\frac{2}{\varepsilon}\right)}$$

$$= \underbrace{\frac{1}{1+\frac{\alpha}{2\pi}\frac{1}{\varepsilon}}}_{Z_2}\ \frac{1}{\not p - m\left(1+\frac{3\alpha}{2\pi}\frac{1}{\varepsilon}\right)}$$

$$\Rightarrow \left.\begin{aligned} \delta Z_2 &= -\frac{\alpha}{2\pi}\frac{1}{\varepsilon} + \text{finite} \\ \delta m &= m\left(\frac{3\alpha}{2\pi}\frac{1}{\varepsilon} + \text{finite}\right) \end{aligned}\right\}$$

works if $\frac{2}{\varepsilon} \leftrightarrow \ln\left(\frac{M^2}{m_\gamma^2}\right)$.

$$\Rightarrow i\frac{\alpha_{EM}}{2\pi}\frac{1}{\varepsilon}(\not p - 4m) + i(\not p\,\delta_2 - \delta_m) = \text{finite}$$

$$\Rightarrow \not p\, i\left[\delta_2 + \frac{\alpha_{EM}}{2\pi}\frac{1}{\varepsilon}\right] - i\left[m\cdot 2\frac{\alpha_{EM}}{\pi}\frac{1}{\varepsilon} + \delta_m\right] = \text{finite}$$

$$\Rightarrow \boxed{\begin{aligned} \delta_2 &= -\frac{\alpha}{2\pi}\frac{1}{\varepsilon} + \text{finite} \\ \delta_m &= -2\frac{\alpha}{\pi}\frac{m}{\varepsilon} + \text{finite} \end{aligned}} \qquad \boxed{\delta_3 = -\frac{2\alpha}{3\pi}\frac{1}{\varepsilon} + \text{finite}}$$

$\delta_1 = \delta_2$ Ward-Takahashi

<u>Problem</u>: while requiring that the sum of diagrams is finite fixed the divergent parts of δ_2, δ_m, it did <u>not</u> fix the constants (finite parts)!

This is not a bug, but a feature: we are free to choose constants in any way we want! $\Rightarrow$ different renorm. schemes

QED "on-shell" renormalization conditions:

(i) $\Sigma(p)\big|_{\not p = m} = 0$ $\quad$ $\dfrac{\partial \Sigma(p)}{\partial \not p}\bigg|_{\not p = m} = 0$

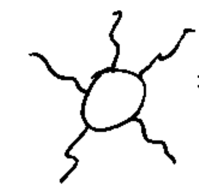
Want $S(p) = \dfrac{i}{\not p - m}$ + finite.

(ii) $\Pi(q^2=0) = 0$ $\quad$ want $D_{\mu\nu}(q) = \dfrac{-i g_{\mu\nu}}{q^2 + i\varepsilon} + (q_\mu q_\nu\text{-terms})$

(iii) $\Gamma^\mu(q=0) = \gamma^\mu$ $\quad$  $= -ie\gamma^\mu$.

$\Rightarrow$ 4 conditions fix $\delta_1, \delta_2, \delta_3$ & δ_m uniquely!

We argued then that there is no other divergent 1-loop graphs in QED

(e.g.  $= 0$,  $= 0$, ... Furry's theorem:

$\mathcal{L}_{QED}$ is invariant under charge conjugation:

$$\begin{cases} \psi_\alpha \to C_{\alpha\beta}\bar\psi_\beta, & C = i\gamma^2\gamma^0, \quad \bar\psi_\alpha \to \psi_\beta C_{\beta\alpha} \\ A_\mu \to -A_\mu & x_\mu \to -x_\mu \end{cases}$$

$\bar\psi\gamma^\mu\psi \xrightarrow{C} -\bar\psi\gamma^\mu\psi$ (can show)

$\Gamma_3 = \langle 0| A_\mu(x) A_\nu(y) A_\rho(z) |0\rangle \xrightarrow{C} -\Gamma_3$, but as $\mathcal{L}_{QED}$ is C-inv.

$\Rightarrow -\Gamma_3 = (\Gamma_3)^C = \Gamma_3 \Rightarrow \Gamma_3 = 0.$)

$\sim \int^{\Lambda} \frac{d^4k}{k^4} \sim \ln \Lambda$ => in fact finite (can show).

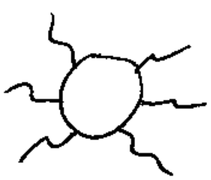
In general can characterize the diagram by its <u>superficial degree of divergence</u>: $D = 4L - P_e - 2P_\gamma$

L = # loops (each loop gives d^4k)

P_e = # of electron propagators (each fermion prop. gives $1/k$)

P_γ = # — photon — (each gives $1/k^2$).

=> the diagram should diverge at most as Λ^D.
(if $D<0$ + subdiagrams => convergent diagram Weinberg's th'm)



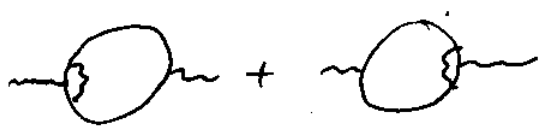
$\sim \Lambda^{4 \cdot 1 - 6} \sim \Lambda^{-2} \sim \frac{1}{\Lambda^2}$ => finite

$L=1,\ P_e=6,\ P_\gamma=0$ (all other multi-leg 1-loops are finite too)

What about multi-loop graphs? One can show that UV divergences are removed by counterterms:

UV-div. only

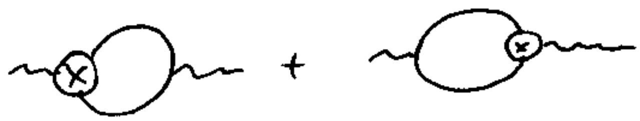
$\approx$



+

counterterm at $O(\alpha^2)$

1st & 2nd terms are removed by



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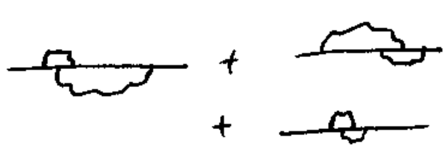
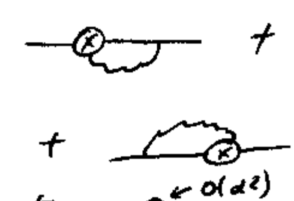


=> removed by



$\leftarrow O(\alpha^2)$

=> QED is renormalizable!

In general one can tell if the theory is renormalizable by dimension of the coupling constant: if $\dim \lambda \overset{!}{=} n$

$\Rightarrow \lambda \sim M^n \Rightarrow$ each λ comes with $\frac{1}{p^n} \Rightarrow$ get $\left(\frac{M}{p}\right)^n$.

$\Rightarrow n > 0 \Rightarrow$ higher orders have less UV divergences than lower orders

$n = 0 \Rightarrow$ higher order graphs are as divergent as lower order ones.

$n < 0 \Rightarrow$ higher order graphs are more divergent than LO ones.

$\Rightarrow$ $n > 0$: <u>super-renormalizable</u> theory.
(e.g. φ^4 in 3-dim)

$n = 0$: <u>renormalizable</u> theory
(φ^4 in 4-dim, QED)

$n < 0$: <u>non-renormalizable</u> theory
(e.g. φ^6 in 4 dim.)