Last time / Reviewed renormalization of QED:

$$\boxed{\begin{aligned}\mathcal{L}_{QED} = &-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}[i\not{\partial} - m]\psi - e\bar{\psi}\gamma^\mu\psi A_\mu \\ &-\tfrac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i\delta_2\not{\partial} - \delta_m]\psi - e\delta_1\bar{\psi}\gamma^\mu\psi A_\mu\end{aligned}}$$

Counterterms $\delta_1, \delta_2, \delta_3, \delta_m$ ~ cancel infinities, defined up to a constant.

~ On-shell renormalization conditions ~ one way of fixing the constants.

~ QED is renormalizable to all orders.

~ in general if $\lambda \sim M^n \Rightarrow$

$n > 0$ super-renormalizable

$n = 0$ renormalizable

$n < 0$ non-renormalizable (bad)

Running of QED Coupling Constant

Renormalized QED Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}[i\not{\partial} - m]\psi - e\bar{\psi}\gamma^\mu \psi A_\mu - \frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu}$$

$$+ \bar{\psi}[i\delta_2 \not{\partial} - \delta_m]\psi - e\delta_1 \bar{\psi}\gamma^\mu\psi A_\mu .$$

Consider dimensional regularization. In d-dimensions $\mathcal{L}$ has dimensions of $M^d \Rightarrow F_{\mu\nu}^2 \sim M^2 A_\mu^2 \sim M^d$

$$\Rightarrow [A_\mu] = M^{\frac{d-2}{2}} \qquad \bar{\psi} m \psi \sim M\psi^2 \sim M^d$$

$$\Rightarrow [\psi] = M^{\frac{d-1}{2}} .$$

Hence $e\bar{\psi}\gamma^\mu\psi A_\mu \sim e\,\psi^2 A_\mu \sim e\, M^{d-1} \cdot M^{\frac{d-2}{2}} \sim M^d$

$$\Rightarrow [e] = M^{\frac{4-d}{2}} = M^{\frac{\varepsilon}{2}} .$$

The coupling becomes dimensionfull! In QED the coupling is dimensionless $\Rightarrow$ to keep it this way replace $e \to e\cdot\mu^{\varepsilon/2}$ with μ some arbitrary momentum scale. We then have

$$\boxed{\begin{aligned}\mathcal{L} = &-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i\not{\partial} - m]\psi - e\mu^{\varepsilon/2}\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} \\ &+ \bar{\psi}[i\delta_2\not{\partial} - \delta_m]\psi - e\mu^{\varepsilon/2}\delta_1\bar{\psi}\gamma^\mu\psi A_\mu .\end{aligned}}$$

$\mu \sim$ renormalization scale.

MS & $\overline{MS}$ Renormalization Conditions

One can impose other renormalization conditions. Each corresponds to a different choice of counterterms $\delta_1, \delta_2, \delta_3, \delta_m$ in QED.

Def. Minimal subtraction (MS) method $\Rightarrow$

$\Rightarrow$ make the counterterms simply remove $\frac{1}{\varepsilon}$ poles of divergent diagrams.

Def. Modified minimal subtraction ($\overline{MS}$) scheme: remove $\left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi)\right)$-terms, as they always come together.

We have in $\overline{MS}$-scheme: $\delta_3^{\overline{MS}} = -\frac{\alpha}{3\pi}\left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right]$.

before: $\Pi_2(q^2) = -\frac{2\alpha\mu^\varepsilon}{\pi}\int_0^1 dx\, x\,(1-x)\left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln(m^2 - x(1-x)q^2)\right] \Rightarrow$

$$\Rightarrow \Pi_{ren}(q^2) = \Pi_2(q^2) - \delta_3 = +\frac{2\alpha}{\pi}\int_0^1 dx\, x\,(1-x)\ln\left[\frac{m^2 - x(1-x)q^2}{\mu^2}\right].$$

$\Rightarrow$ finite. (If we pick $\mu = m \Rightarrow$ get the same answer as in "on-shell" case)

$\Rightarrow$ similarly adjust δ_2 & δ_m to make

$\Sigma(p) = \Sigma_2(p) - \delta_2 \not{p} + \delta_m$ finite by removing

$\left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi)\right)$-terms.

$\Rightarrow$ finally $\delta_1 = \delta_2$ as before.

(If you insist that the problem with dimensionful e goes away as $\varepsilon \to 0$, note that $e = e_0 \frac{Z_2}{Z_1} Z_3^{1/2} = e_0 Z_3^{1/2} =$

$= e_0 (1+\delta_3)^{1/2} = e_0 \left[1 - \frac{\alpha}{3\pi} \left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln m^2 \right] \right]^{1/2}$ in the "on-shell" scheme. Get dimension of $\ln M$, whatever this is.)

We replace $e \to e \mu^{\varepsilon/2}$ $\Rightarrow$ the new coupling is

$$\boxed{e \mu^{\varepsilon/2} = e_0 \frac{Z_2}{Z_1} Z_3^{1/2}}$$

now e is dimensionless

In QED $Z_2 = Z_1$ $\Rightarrow$ $\boxed{e^2 = e_0^2 \mu^{-\varepsilon} Z_3}$.

We get: $e^2 = e_0^2 \mu^{-\varepsilon} Z_3 = e_0^2 \mu^{-\varepsilon} [1+\delta_3] =$

$$= e_0^2 \mu^{-\varepsilon} \left[1 - \frac{\alpha}{3\pi} \left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln m^2 \right] \right] \approx$$

$$\approx e_0^2 \mu^{-\varepsilon} \left[1 - \frac{\alpha}{3\pi} \cdot \frac{2}{\varepsilon} + \text{finite} \right]$$

$\Rightarrow$ rewrite $\alpha = \alpha_0 \mu^{-\varepsilon} \left[1 - \frac{\alpha}{3\pi} \frac{2}{\varepsilon} \right]$ $\quad \left(\alpha = \frac{e^2}{4\pi} \right)$

as $\alpha_0 = \alpha \mu^{\varepsilon} \left[1 + \frac{\alpha}{3\pi} \cdot \frac{2}{\varepsilon} \right]$.

Now, α_0 is μ-independent (bare coupling, does not "know" about new scale μ). We then write $\alpha = \alpha_\mu$ and

$$0 = \mu^2 \frac{d\alpha_0}{d\mu^2} = \mu^2 \frac{d}{d\mu^2}\left\{ (\mu^2)^{\varepsilon/2} \cdot \alpha_\mu \cdot \left[1 + \frac{\alpha_\mu}{3\pi} \cdot \frac{2}{\varepsilon} \right] \right\} =$$

$$= \mu^\varepsilon \frac{\varepsilon}{2} \alpha_\mu \left[1 + \frac{\alpha_\mu}{3\pi} \frac{2}{\varepsilon} \right] + \mu^2 \frac{d\alpha_\mu}{d\mu^2} \cdot \mu^\varepsilon \left[1 + \frac{\alpha_\mu}{3\pi} \frac{2}{\varepsilon} \right] + \mu^\varepsilon \alpha_\mu \cdot \frac{2}{\varepsilon} \frac{1}{3\pi} \cdot$$

$$\cdot \mu^2 \frac{d\alpha_\mu}{d\mu^2} \Rightarrow \mu^2 \frac{d\alpha_\mu}{d\mu^2} \left[1 + 2 \frac{\alpha_\mu}{3\pi} \frac{2}{\varepsilon} \right] = -\frac{\varepsilon}{2} \alpha_\mu \left[1 + \frac{\alpha_\mu}{3\pi} \frac{2}{\varepsilon} \right]$$

$$\Rightarrow \mu^2 \frac{d\alpha_\mu}{d\mu^2} = -\frac{\varepsilon}{2} \alpha_\mu \left[1 - \frac{\alpha_\mu}{3\pi} \frac{2}{\varepsilon} + o(\alpha_\mu^2) \right]$$

$$\Rightarrow \mu^2 \frac{d\alpha}{d\mu^2} = -\frac{\varepsilon}{2} \alpha_\mu + \frac{\alpha_\mu^2}{3\pi}$$ ⇒ take $\varepsilon \to 0$ limit ⇒

$$\Rightarrow \boxed{\mu^2 \frac{d\alpha_\mu}{d\mu^2} = \frac{\alpha_\mu^2}{3\pi}}$$ ← renormalization group (RG) equation

Def. Beta-function of a theory: $\boxed{\beta(\alpha) \equiv \mu^2 \frac{d\alpha}{d\mu^2}}$

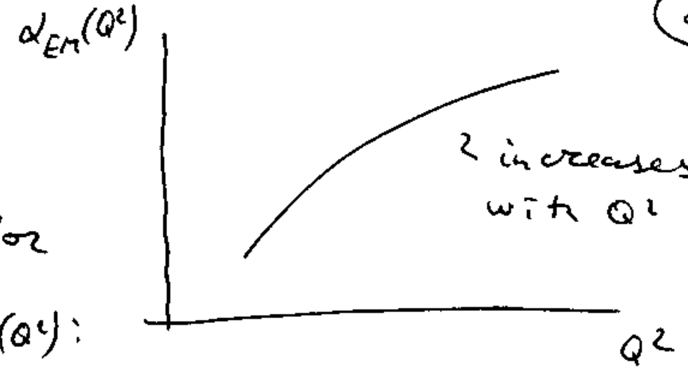
In QED the beta-function is $\boxed{\beta_{QED}(\alpha) = \frac{\alpha^2}{3\pi}}$.

Solve $\frac{d\alpha}{d\ln\mu^2} = \frac{\alpha^2}{3\pi} \Rightarrow \frac{d\alpha}{\alpha^2} = \frac{1}{3\pi} d\ln\mu^2 \Rightarrow$

$$\Rightarrow -\frac{1}{\alpha}\Bigg|_{\alpha_\mu}^{\alpha(Q^2)} = \frac{1}{3\pi} \ln\mu^2 \Bigg|_{\mu^2}^{Q^2} \Rightarrow -\frac{1}{\alpha(Q^2)} + \frac{1}{\alpha_\mu} = \frac{1}{3\pi} \ln\left(\frac{Q^2}{\mu^2}\right)$$

$$\Rightarrow \boxed{\alpha(Q^2) = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi} \ln\left(\frac{Q^2}{\mu^2}\right)}}$$ ← running of QED coupling (like $\alpha_{eff}(Q^2)$ before).

We can plot the coupling:



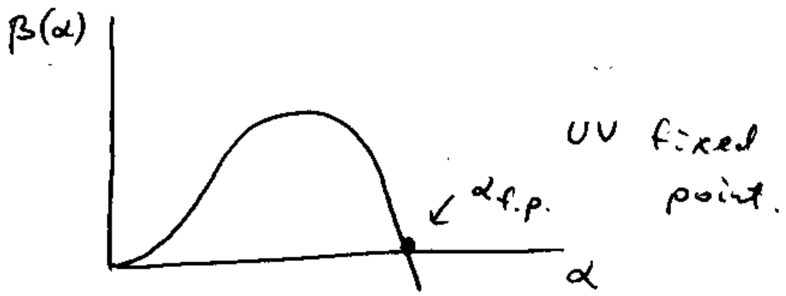
Note a problem: denominator may become 0, giving ∞ $\alpha(Q^2)$:

$$1 = \frac{\alpha_\mu}{3\pi} \ln\left(\frac{\Lambda^2}{\mu^2}\right) \Rightarrow \Lambda^2 = \mu^2 e^{3\pi/\alpha_\mu}$$

$$\Rightarrow \boxed{Q^2 = \mu^2 e^{3\pi/\alpha}}$$ ~ Landau singularity

(QED is incomplete, gets modified in the UV)

The full QED beta-function may look like



Then