Last time | Running of QED Coupling Constant

Noticed that in d-dimensions QED coupling e is dimensionfull => replace $e \to e\mu^{\varepsilon/2}$ to make it dimensionless. => $e^2 = e_0^2 \mu^{-\varepsilon} Z_3 \Rightarrow \alpha_0 = \frac{\alpha \mu^{\varepsilon}}{Z_3}$

=> α_0 is μ-independent => $0 = \mu^2 \frac{d}{d\mu^2}\alpha_0 = \mu^2 \frac{d}{d\mu^2}\left(\frac{\alpha\mu^{\varepsilon}}{Z_3}\right)$

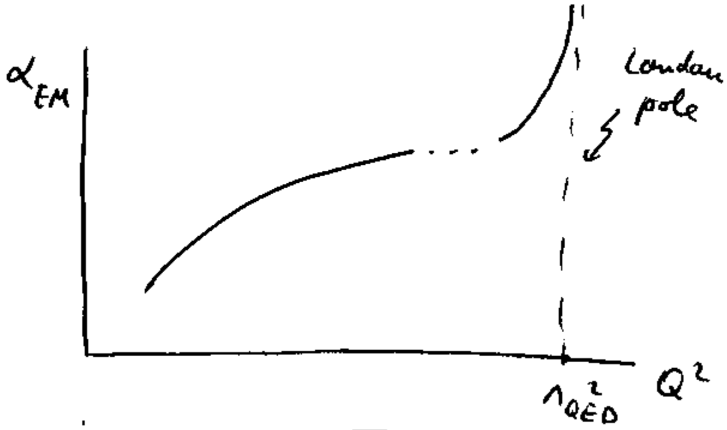
=> found that $\boxed{\mu^2 \frac{d\alpha_\mu}{d\mu^2} = \frac{\alpha_\mu^2}{3\pi}}$.

Def. Beta-function : $\boxed{\beta(\alpha) \equiv \mu^2 \frac{d\alpha_\mu}{d\mu^2}}$

=> $\boxed{\beta_{QED}(\alpha) = \frac{\alpha^2}{3\pi}}$ ~ QED beta-function

$$\boxed{\alpha(Q^2) = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi}\ln\frac{Q^2}{\mu^2}}}$$

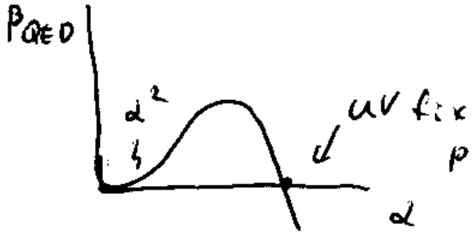
QED running coupling



$$\boxed{\alpha_{EM}(Q^2) = \frac{1}{\frac{1}{3\pi}\ln\frac{\Lambda_{QED}^2}{Q^2}}}$$ with $\boxed{\Lambda_{QED}^2 = \mu^2 e^{3\pi/\alpha_\mu}}$

Landau singularity

It may be that



=>

Def. Minimal subtraction (MS) scheme:

choose counterterms δ's to simply remove $\sim \frac{1}{\varepsilon}$ singularities in dim.reg.

Def. Modified MS ($\overline{MS}$) scheme:

— 1 — to remove $\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right)$-terms.

Renormalization of φ^4 Theory.

Start with bare Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_0\partial^\mu\varphi^0 - \frac{m_0^2}{2}\varphi_0^2 - \frac{\lambda_0}{4!}\varphi_0^4.$$

$\boxed{\text{Define}}$ physical field φ by $\varphi_0 = \sqrt{Z}\,\varphi$

$$\Rightarrow \mathcal{L} = \frac{Z}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{Z m_0^2}{2}\varphi^2 - \frac{\lambda_0}{4!}\varphi^4\cdot Z^2$$

⇒ Can define physical mass m & coupling λ & write

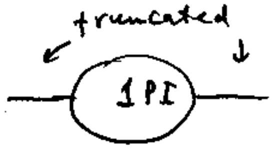
← again inserted momentum scale to make λ dimensionless.

$$\boxed{\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda\mu^\varepsilon}{4!}\varphi^4 \\ &+ \frac{1}{2}\delta_Z\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\delta_m\varphi^2 - \frac{\delta_\lambda}{4!}\varphi^4\end{aligned}}$$

Here $\boxed{\delta_Z = Z-1,\quad \delta_m = m_0^2 Z - m^2,\quad \delta_\lambda = \lambda_0 Z^2 - \lambda\mu^\varepsilon}$.

Again, just like in QED have 3 counterterms, but now only have 3 constants.

Demand "on-shell" renormalization conditions:

$-i\,\Sigma_{\text{pert}}(p^2) =$  ⇒ renormalized propagator is

$$\frac{i}{k^2-m^2} + \frac{i}{k^2-m^2}(-i\Sigma)\frac{i}{k^2-m^2} + \ldots = \frac{i}{k^2-m^2-\Sigma(k^2)}$$

⇒ want it to be $\frac{i}{k^2-m^2}$ after renormalization

=> demand that

$$\boxed{\Sigma(p^2=m^2)=0\ ,\quad \left.\frac{\partial\Sigma}{\partial p^2}\right|_{p^2=m^2}=0}.$$

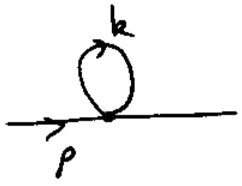
We may define the coupling by requiring that renormalized truncated 4-pt. Green function is

$$\boxed{\Gamma^4(s=t=u=M^2)=-i\lambda}\ ,\quad \Gamma^4(s,t,u)=$$ 

.

or $\Gamma^4(s=4m^2,\ t=u=0)=-i\lambda$, etc.

Let's regularize the theory first (using dim.reg.):



$$\Rightarrow\ -i\Sigma_2(p^2)=-\underset{\substack{\uparrow\\ \text{symmetry}\\ \text{factor}}}{\frac{i\lambda}{2!}}\mu^\varepsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2+i\varepsilon}=\Big|\begin{smallmatrix}\text{Wick}\\ \text{rotation}\end{smallmatrix}$$

$$=\frac{-i\lambda}{2!}\mu^\varepsilon\cancel{(-i)}\cdot\cancel{i}\cdot\int\frac{d^dk_E}{(2\pi)^d}\,\frac{1}{k_E^2+m^2}=-\frac{i\lambda\mu^\varepsilon}{2}\cdot\int\frac{d\Omega_d}{(2\pi)^d}\cdot\int_0^\infty\frac{dl_E\,k_E^{d-1}}{k_E^2+m^2}$$

$$=\Bigg|\ \begin{matrix}\text{In general}\\ \boxed{\int\frac{d^dk_E}{(2\pi)^d}\,\frac{1}{\left[k_E^2+\Lambda^2\right]^n}=\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{\Lambda^2}\right)^{n-\frac{d}{2}}}\end{matrix}\ \Rightarrow\text{for us } n=1$$

$$=\frac{-i\lambda}{2}\mu^\varepsilon\,\frac{1}{(4\pi)^{d/2}}\,\Gamma\left(1-\frac{d}{2}\right)\cdot(m^2)^{\frac{d}{2}-1}=\Big|\ d=4-\varepsilon\qquad=$$

$$=-i\,\frac{\lambda}{2}\,\mu^\varepsilon\,\frac{1}{(4\pi)^{2-\frac{\varepsilon}{2}}}\,\Gamma\left(-1+\frac{\varepsilon}{2}\right)(m^2)^{1-\frac{\varepsilon}{2}}$$

$$\Gamma\left(-1+\frac{\varepsilon}{2}\right)\cdot\left(-1+\frac{\varepsilon}{2}\right)=\Gamma\left(\frac{\varepsilon}{2}\right)=\frac{2}{\varepsilon}-\gamma+o(\varepsilon)$$

$$\Rightarrow \Gamma\left(-1+\frac{\varepsilon}{2}\right) = (-1)\left(1+\frac{\varepsilon}{2}\right)\left(\frac{2}{\varepsilon}-\gamma+\dots\right) = (-1)\left(\frac{2}{\varepsilon}-\gamma+1+\dots\right)$$

$$-i\Sigma_2 = -i\,\frac{\lambda}{2}\left(1+\varepsilon\ln\mu\right)\frac{1}{(4\pi)^2}\cdot\left(1+\frac{\varepsilon}{2}\ln 4\pi\right)(-1)\left(\frac{2}{\varepsilon}-\gamma+1\right)\cdot m^2$$

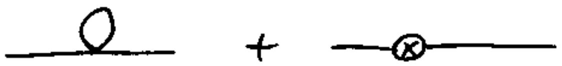
$$\cdot\left(1-\frac{\varepsilon}{2}\ln m^2\right) = +i\,\frac{\lambda m^2}{32\pi^2}\left(1+\frac{\varepsilon}{2}\ln\frac{\mu^2}{m^2}+\frac{\varepsilon}{2}\ln 4\pi\right)\left(\frac{2}{\varepsilon}-\gamma+1\right)$$

$$= i\,\frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\varepsilon}+\ln\frac{\mu^2}{m^2}+\ln(4\pi)-\gamma+1+\dots\right] \Rightarrow$$

$$\Rightarrow \boxed{-i\Sigma_2 = i\,\frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\varepsilon}+\ln\left(\frac{\mu^2}{m^2}\right)+\ln(4\pi)-\gamma+1+O(\varepsilon)\right]}$$

The counterterm is $-\otimes- = i\left(p^2\delta_Z-\delta_m\right)$

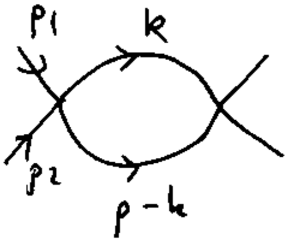
$$\Rightarrow -i\Sigma(p^2) = -i\Sigma_2 + i\left(p^2\delta_Z-\delta_m\right).$$



Imposing $\left.\frac{\partial\Sigma}{\partial p^2}\right|_{p^2=m^2} = 0$ gives $\boxed{\delta_Z = 0}$ as Σ_2 is independent of p^2. $\Rightarrow \Sigma(p^2) = \Sigma_2 + \delta_m$

$\Rightarrow$ as $\Sigma(p^2=m^2)=0 \Rightarrow \boxed{\delta_m = -\Sigma_2}$

such that $\boxed{\Sigma(p^2)=0}$ at order λ.



$\Rightarrow$ contribution to $\Gamma^4(s,t,u)$ is

$$\frac{(-i\lambda)^2}{2!}\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\varepsilon}\frac{i}{(p-k)^2-m^2+i\varepsilon} =$$

$p = p_1+p_2$

$$=\frac{\lambda^2}{2}\mu^{2\varepsilon}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left[(1-x)(k^2-m^2)+x[(p-k)^2-m^2]\right]^2}$$

$$=\frac{\lambda^2}{2}\mu^{2\varepsilon}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left[k^2-2xp\cdot k+xp^2-m^2\right]^2} = \Big|\ \ell = k-xp$$

$$=\frac{\lambda^2}{2}\mu^{2\varepsilon}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{\left[\ell^2+x(1-x)p^2-m^2\right]^2} =$$

| Wick rotation
| $i\ell^0_E = \ell^0$
| $-\ell_E^2 = \ell^2$

$$=\frac{\lambda^2}{2}\mu^{2\varepsilon}\cdot i\int_0^1 dx\int\frac{d^d\ell_E}{(2\pi)^d}\frac{1}{\left(\ell_E^2+m^2-x(1-x)p^2\right]^2} =$$

| using the same f-la as for 
| $n=2$

$$=\frac{\lambda^2}{2}\mu^{2\varepsilon}\cdot i\cdot\int_0^1 dx\cdot\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{1}\left[m^2-x(1-x)p^2\right]^{\frac{d}{2}-2} =$$

| $\varepsilon = 4-d$
| $d = 4-\varepsilon$

$$=\frac{\lambda^2}{2}\frac{\mu^{2\varepsilon}}{(4\pi)^2}\cdot i\int_0^1 dx\,(4\pi)^{\frac{\varepsilon}{2}}\,\Gamma\left(\frac{\varepsilon}{2}\right)\left[m^2-x(1-x)p^2\right]^{-\frac{\varepsilon}{2}} =$$

$$=i\frac{\lambda^2\mu^\varepsilon}{32\pi^2}\int_0^1 dx\left(1+\frac{\varepsilon}{2}\ln\mu^2+\frac{\varepsilon}{2}\ln 4\pi\right)\left(\frac{2}{\varepsilon}-\gamma\right)\left(1-\frac{\varepsilon}{2}\ln\left[m^2-x(1-x)p^2\right]\right)$$

$$=i\frac{\lambda^2\mu^\varepsilon}{32\pi^2}\int_0^1 dx\left[\frac{2}{\varepsilon}-\gamma+\ln 4\pi-\ln\left(\frac{m^2-x(1-x)p^2}{\mu^2}\right)\right],\quad p^2 = s.$$

Adding s, t, u channels: 

gives

$$\boxed{\begin{aligned}\Gamma^4_{reg.}(s,t,u) = -i\lambda\mu^{\varepsilon}\Big\{ 1 + \frac{\lambda}{32\pi^2}\int_0^1 dx\Big[\ln\left(\frac{m^2-x(1-x)s}{\mu^2}\right)+\gamma-\ln 4\pi-\frac{2}{\varepsilon}+ \\ +(s\leftrightarrow t)+(s\leftrightarrow u)\Big\}\end{aligned}}$$

The counterterm is  $= -i\,\delta_\lambda$

$$\Rightarrow \quad \Gamma^4 = \Gamma^4_{reg.} - i\,\delta_\lambda$$

Requiring that $\Gamma^4(s=t=u=-\mu^2)=-i\lambda$ fixes δ_λ. One may use other conditions. One gets: (note that δ_λ is dimensionfull!)

$$+i\overset{s,t,u}{3}\frac{\lambda^2\mu^\varepsilon}{32\pi^2}\frac{2}{\varepsilon}{}^{+\text{ finite}} - i\,\delta_\lambda = 0 \Rightarrow \boxed{\delta_\lambda = \frac{3\lambda^2\mu^\varepsilon}{16\pi^2}\cdot\frac{1}{\varepsilon}+\text{finite}}.$$

at $o(\lambda)$: $\delta_Z = 0 \Rightarrow Z = 1 \Rightarrow \delta_\lambda = \lambda_0 - \lambda\mu^\varepsilon \Rightarrow \lambda_0 = \delta_\lambda + \lambda\mu^\varepsilon$

$$\Rightarrow \lambda_0 = \lambda\mu^\varepsilon\left(1+\frac{3\lambda}{16\pi^2}\frac{1}{\varepsilon}+\text{finite}\right) \Rightarrow 0 = \mu^2\frac{d}{d\mu^2}\lambda_0 =$$

$$= \mu^2\frac{d}{d\mu^2}\left\{\lambda(\mu^2)^{\varepsilon/2}\left[1+\frac{3\lambda}{16\pi^2}\frac{1}{\varepsilon}+\dots\right]\right\} = \left(\mu^2\frac{d\lambda}{d\mu^2}\right)\mu^\varepsilon\left[1+\frac{3\lambda}{16\pi^2}\frac{1}{\varepsilon}\right]$$

$$+\frac{\varepsilon}{2}\lambda(\mu^2)^{\varepsilon/2}\left[1+\frac{3\lambda}{16\pi^2}\frac{1}{\varepsilon}\right]+\lambda\mu^\varepsilon\frac{3}{16\pi^2\varepsilon}\mu^2\frac{d\lambda}{d\mu^2} = 0$$

$$\Rightarrow \left(\mu^2\frac{d\lambda}{d\mu^2}\right)\cancel{\mu^\varepsilon}\left[1+\frac{3\lambda}{8\pi^2}\frac{1}{\varepsilon}\right] = -\frac{\varepsilon}{2}\lambda\cancel{\mu^\varepsilon}\left[1+\frac{3\lambda}{16\pi^2}\frac{1}{\varepsilon}\right]$$

$$\Rightarrow \quad \mu^2 \frac{d\lambda}{d\mu^2} = -\frac{\varepsilon}{2}\lambda\left[1 - \frac{3\lambda}{16\pi^2}\frac{1}{\varepsilon} + \dots\right] = -\frac{\varepsilon}{2}\lambda + \frac{3\lambda^2}{32\pi^2} + \mathcal{O}(\varepsilon)$$

$$\Rightarrow \quad \boxed{\mu^2 \frac{d\lambda}{d\mu^2} = \frac{3\lambda^2}{32\pi^2}} \equiv \beta(\lambda)$$ ~ beta-function of φ^4 theory.

renormalization group eq'n.

$$\Rightarrow \quad \frac{d\lambda}{\lambda^2} = \frac{3}{32\pi^2}\,\frac{d\mu^2}{\mu^2} \Rightarrow -\frac{1}{\lambda(Q)} + \frac{1}{\lambda_\mu} = \frac{3}{32\pi^2}\ln\left(Q^2/\mu_0^2\right)$$

$$\Rightarrow \quad \boxed{\lambda(Q^2) = \frac{\lambda_\mu}{1 - \lambda_\mu \cdot \frac{3}{32\pi^2} \cdot \ln\left(\frac{Q^2}{\mu^2}\right)}}$$

similar to QED.

The Callan-Symanzik Equation.

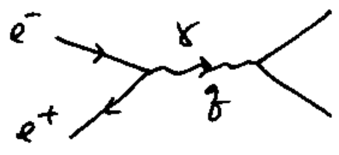
Consider an observable which is dimensionless and depends on one momentum scale Q^2 only.

← coupling constant

$$M = M(Q^2, \mu^2, \alpha_\mu)$$

↖ renormalization scale

Example| $e^+e^- \to$ hadrons



$\sigma_{tot}(Q^2)$ ~ total cross-section integrated over all t/u => depends only on $s = Q^2 = q_\mu q^\mu$.

"s

$\Rightarrow Q^2\sigma_{tot}(Q^2)$ is dim-less and depends on Q^2 only!

=> in general would have $M = M(Q^2, \alpha_\mu, \mu)$

where $\boxed{\alpha_\mu = \frac{g_\mu^2}{4\pi}}$

=> assume that M is dimensionless => $M = M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right)$.

But: no physical observable should depend on μ!

$$\Rightarrow \boxed{\mu^2 \frac{d}{d\mu^2} M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0}$$

$$\Rightarrow \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{d\alpha_\mu}{d\mu^2} \frac{\partial}{\partial \alpha_\mu}\right] M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0$$

Def. Beta-function of the theory: $\boxed{\beta(\alpha_\mu) = \mu^2 \frac{d\alpha_\mu}{d\mu^2}}$.

$\beta(\alpha_\mu)$ is dimensionless => can not depend on μ explicitly, μ-dependence comes in through α_μ only

$$\Rightarrow \boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right] M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0}$$

renormalization group equation ((Callan, Symanzik '70))

~ tells how things change with the changing momentum scale / distance resolution

=> equivalently $\left[-Q^2 \frac{\partial}{\partial Q^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right] M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0$.