Last time | Renormalized φ^4 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda\mu^\varepsilon}{4!}\varphi^4 + \frac{1}{2}\delta_Z\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\delta_m\varphi^2 - \frac{\delta_\lambda}{4!}\varphi^4$$

To find δ_Z & δ_m we calculated  using dim. reg.

Got: $\delta_Z = 0$ (on-shell, MS, $\overline{MS}$)

$$\delta_m^{\text{on-shell}} = \frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right) - \gamma + \ln 4\pi + 1\right]$$

$$\delta_m^{MS} = \frac{\lambda m^2}{32\pi^2}\,\frac{2}{\varepsilon}$$

$$\delta_m^{\overline{MS}} = \frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right]$$

To find δ_λ we calculated  using dim. reg.

$\Rightarrow$ got $\boxed{\delta_\lambda = \frac{3\lambda^2\mu^\varepsilon}{16\pi^2}\frac{1}{\varepsilon} + \text{finite}}$

^ depends on scheme

$\Rightarrow \quad \delta_\lambda = \lambda_0 Z^2 - \lambda\mu^\varepsilon \Rightarrow$ as $\delta_Z = 0 \Rightarrow Z = 1 \Rightarrow$

$$\lambda_0 = \delta_\lambda + \lambda\mu^\varepsilon \quad \Rightarrow \quad 0 = \mu^2\frac{d}{d\mu^2}\lambda_0 = \mu^2\frac{d}{d\mu^2}\left[\delta_\lambda + \lambda\mu^\varepsilon\right]$$

$\Rightarrow$ we found $\boxed{\beta(\lambda) \equiv \mu^2\frac{d\lambda}{d\mu^2} = \frac{3\lambda^2}{32\pi^2}}$

beta-function of φ^4 theory!

The Callan-Symanzik Equation (cont'd)

$$\mathcal{M} = \mathcal{M}(Q^2, \mu^2, \alpha_\mu) = \mathcal{M}\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right)$$

~ dimensionless observable, depends only on one physical momentum scale Q^2.

$$\boxed{\mu^2 \frac{d}{d\mu^2} \mathcal{M}\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = 0}$$ ~ independent of renormalization scale μ

$$\Rightarrow \boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right] \mathcal{M}\left(Q^2/\mu^2, \alpha_\mu\right) = 0}$$

Callan-Symanzik equation '70

$$\beta(\alpha_\mu) = \mu^2 \frac{d\alpha_\mu}{d\mu^2}$$ beta-function

To solve the renormalization group (RG) equation define $\rho(\alpha_\mu) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')}$

α_0 ← arbitrary cutoff

Def. Running Coupling by:

$$\boxed{\alpha(Q^2) \equiv \rho^{-1}\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right)} \qquad \rho^{-1} \sim \text{inverse function}$$

⇒ note that

(i) $\alpha(\mu^2) = \alpha_\mu$

(ii) $\left[\mu^2 \frac{\partial}{\partial\mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial\alpha_\mu}\right]\alpha(Q^2) = 0$

[Item (ii) is true because $\left[\mu^2 \frac{\partial}{\partial\mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial\alpha_\mu}\right]\cdot$

$$\cdot\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right) = -1 + \beta(\alpha_\mu)\underbrace{\frac{\partial\rho(\alpha_\mu)}{\partial\alpha_\mu}}_{1/\beta(\alpha_\mu) \text{ by definition}} = 0$$

as $M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right)$ does not depend on μ we can put $\mu = Q$ and get:

$$M\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = M\left(\frac{Q^2}{\mu^2}, \alpha(\mu^2)\right) \overset{\mu^2 \to Q^2}{=} M(1, \alpha(Q^2)) = \mathcal{M}(\alpha(Q^2))$$

⇒ any $\mathcal{M}$ which is a function of $\alpha(Q^2)$ only satisfies the RG equation

Let's find $\alpha(Q^2)$ and prove that this is the same coupling that we had before: to find $\alpha(Q^2)$ all we need is the beta-function $\beta(\alpha_\mu)$.

Usually in perturbation theory one gets:

$$\boxed{\beta(\alpha) = \beta_2 \alpha^2 + \beta_3 \alpha^3 + \ldots}$$

β_2 ~ scheme independent

β_3 ~ depends on renormalization scheme (MS, $\overline{\text{MS}}$, "on-shell", etc.)

$\Rightarrow$ all higher β_i's $(\beta_4, \beta_5, \ldots)$ are also scheme-dependent.

In QED we showed that $\beta_2^{QED} = \frac{1}{3\pi}$.

For φ^4 theory, if $\alpha \leftrightarrow \lambda$ $\Rightarrow$ $\beta_2^{\varphi^4} = \frac{3}{32\pi^2}$.

Keep β_2 only ($\alpha \ll 1 \Rightarrow$ drop higher orders for now):

$$\beta(\alpha) = \beta_2 \alpha^2 \Rightarrow \rho(\alpha_\mu) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha}{\beta(\alpha)} = \frac{1}{\beta_2} \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha}{\alpha^2} =$$

$$= \frac{1}{\beta_2}\left(-\frac{1}{\alpha_\mu} + \frac{1}{\alpha_0}\right) \Rightarrow \rho(\alpha) = -\frac{1}{\beta_2}\left(\frac{1}{\alpha} - \frac{1}{\alpha_0}\right)$$

The inverse function: $\rho(\alpha) = w \Rightarrow \alpha = \rho^{-1}(w)$

$$\Rightarrow \; -\frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right) = w \;\Rightarrow\; \frac{1}{\alpha} = \frac{1}{\alpha_0} - \beta_2 w$$

$$\Rightarrow \; \alpha = \rho^{-1}(w) = \frac{1}{\frac{1}{\alpha_0} - \beta_2 w} \;\Rightarrow\; \frac{\alpha_0}{1 - \beta_2 \alpha_0 w} = \rho^{-1}(w),$$

$$\alpha(Q^2) = \rho^{-1}\left(\ln\frac{Q^2}{\mu^2} + \rho(\alpha_\mu)\right) = \rho^{-1}\left(\ln\frac{Q^2}{\mu^2} - \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right)\right)$$

$$= \frac{\alpha_0}{1 - \beta_2\alpha_0\left(\ln\frac{Q^2}{\mu^2} - \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right)\right)} = \frac{\alpha_0}{\cancel{1} - \beta_2\alpha_0 \ln\frac{Q^2}{\mu^2} + \frac{\alpha_0}{\alpha_\mu} - \cancel{1}}$$

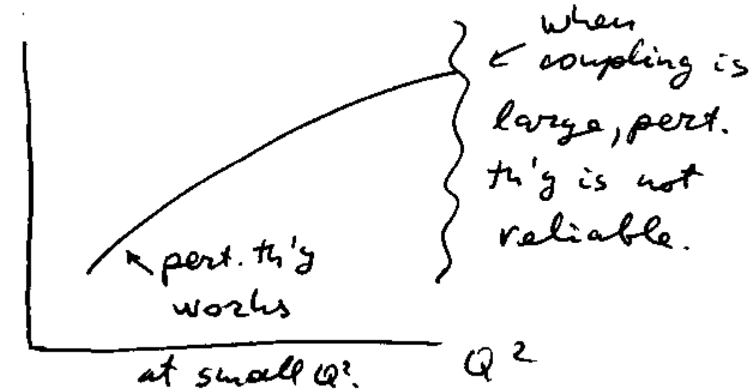
$$= \frac{1}{\frac{1}{\alpha_\mu} - \beta_2 \ln\frac{Q^2}{\mu^2}} = \frac{\alpha_\mu}{1 - \alpha_\mu\beta_2 \ln\frac{Q^2}{\mu^2}}$$

$$\Rightarrow \; \boxed{\alpha(Q^2) = \frac{\alpha_\mu}{1 - \alpha_\mu \beta_2 \ln\frac{Q^2}{\mu^2}}}$$ one-loop running coupling in a gauge theory

Cf. put $\beta_2^{QED} = \frac{1}{3\pi}$ & get QED running coupling

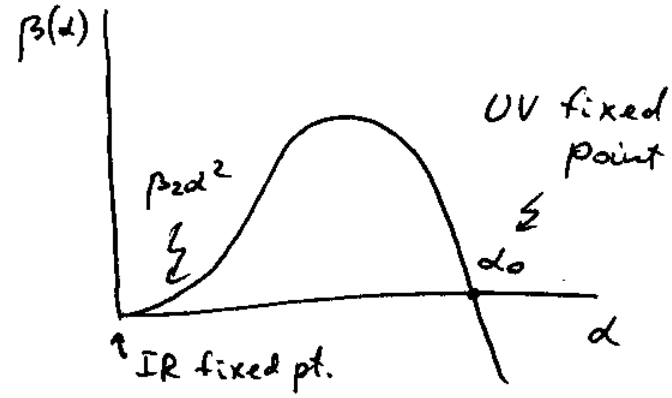
put $\beta_2^{\varphi^4} = \frac{3}{32\pi^2}$ and swap $\alpha \leftrightarrow \lambda$ $\Rightarrow$ get φ^4 running coupling.

small $Q^2 \Leftrightarrow$ large distances



What does the full $\beta(\alpha)$ may look like?

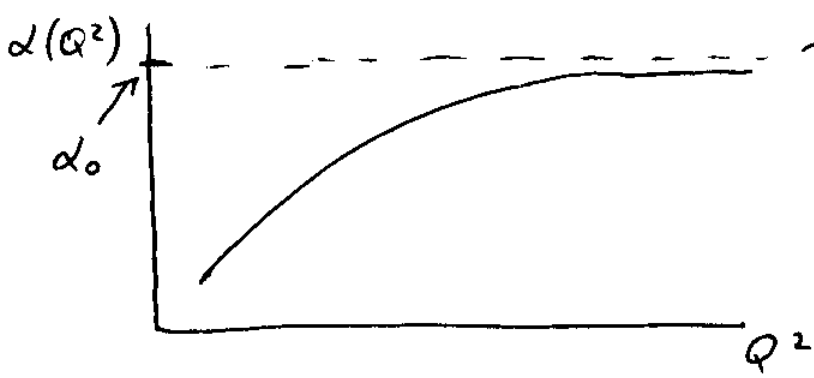
One possibility is this:



As $\mu^2 \dfrac{d\alpha_\mu}{d\mu^2} = \beta(\alpha_\mu) \Rightarrow$

$\Rightarrow$ at $\alpha_\mu = \alpha_0 \Rightarrow \beta(\alpha_0) = 0$

$\Rightarrow \mu^2 \dfrac{d\alpha_\mu}{d\mu^2} = 0 \Rightarrow \alpha_\mu$ becomes a constant:



~ "fixed point" ~ coupling is "frozen" at α_0
UV ~ happens at large-Q^2.

<u>Renormalization group: general discussion.</u>

Consider renormalized n-point Green function:

$$G^{(n)}(x_1, \ldots, x_n) = \langle \Psi_0 | \, T \, \varphi(x_1) \ldots \varphi(x_n) | \Psi_0 \rangle$$

physical fields.

In momentum space get

$$G^{(n)}(p_1, \ldots, p_n) = \int d^4x_1 \ldots d^4x_n \, e^{ip_1 \cdot x_1 + \ldots + ip_n \cdot x_n} \cdot G^{(n)}(x_1, \ldots, x_n).$$

Renormalization "group" is <u>defined</u> by group

transformations: $\mu \to \lambda\mu$, $\lambda = \forall$ parameter.

this is simply a rescaling of μ.

Physical observables $\langle p_1 \dots p_{n-2} | S | k_1 k_2 \rangle \sim Z^{-n/2} \cdot G^{(n)} \Rightarrow$ indep. of $\mu^2 \Rightarrow$

Now, in general $\left(Z^{-n/2} G^{(n)}\right)$ should not depend on μ:

$$\boxed{\mu^2 \frac{d}{d\mu^2}\left[Z^{-n/2} G^{(n)}(p_1, \dots, p_n)\right] = 0}.$$

However, $G^{(n)}$ has in it functions of μ:

$\alpha_\mu \sim$ coupling constant

$m_\mu \sim$ renormalized mass (e.g. $\delta m = m_0^2 Z - m_\mu^2$)

$\Big(\varphi(x) = \frac{1}{\sqrt{Z}} \varphi_0(x) \Rightarrow Z$ depends on $\mu \Rightarrow$

$\varphi(x)$ ~ physical field, $\varphi_0(x)$ ~ bare field

$\Rightarrow \langle p_1 \dots p_{n-2} | S | k_1 k_2 \rangle \sim Z^{-n/2} G^{(n)}$

LSZ reduction f-la $\Big)$

$$\Rightarrow 0 = \mu^2 \frac{d}{d\mu^2}\left[G^{(n)} \underbrace{Z^{-n/2}}_{Z^{-n/2}\,\checkmark}\right] = \left[\mu^2 \frac{\partial}{\partial\mu^2} + \mu^2 \frac{d\alpha_\mu}{d\mu^2}\frac{\partial}{\partial\alpha_\mu} + \frac{\mu^2}{Z^{-n/2}}\frac{dZ^{-n/2}}{d\mu^2} + \mu^2 \frac{dm_\mu^2}{d\mu^2}\frac{\partial}{\partial m_\mu^2}\right] G^{(n)}.$$

Define

$$\beta(\alpha_\mu) = \mu^2 \frac{d\alpha_\mu}{d\mu^2} \sim \text{as before}$$

$$\gamma(\alpha_\mu) = \mu^2 \frac{d \ln\sqrt{Z}}{d\mu^2} \sim \text{anomalous dimension}$$

$$\gamma_m(\alpha_\mu) = \frac{1}{m_\mu^2}\mu^2 \frac{dm_\mu^2}{d\mu^2}.$$

as $z^{n/2}\mu^2 \dfrac{d z^{-n/2}}{d\mu^2} = z^{n/2}\cdot\left(-\frac{n}{2}\right) z^{-\frac{n}{2}-1}\mu^2\dfrac{dz}{d\mu^2} = -\dfrac{n}{2}\dfrac{1}{z}\cdot$

$\cdot\, \mu^2 \dfrac{dz}{d\mu^2}$ and $\mu^2 \dfrac{d\ln\sqrt{z}}{d\mu^2} = \dfrac{1}{2}\,\dfrac{1}{z}\,\mu^2\dfrac{dz}{d\mu^2} = \gamma(\alpha_\mu)$

$$\Rightarrow z^{n/2}\mu^2\frac{dz^{-n/2}}{d\mu^2} = -n\,\gamma(\alpha_\mu)$$

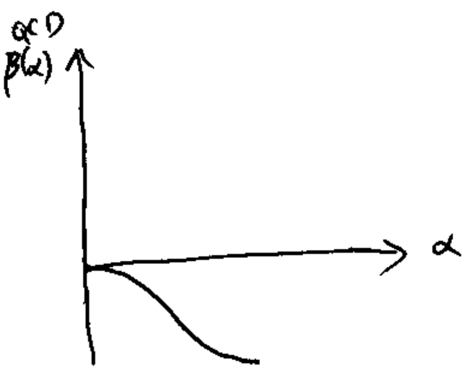
$$\Rightarrow \boxed{\left[\mu^2\frac{\partial}{\partial\mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial\alpha_\mu} - n\,\gamma(\alpha_\mu) + m_\mu^2\gamma_m(\alpha_\mu)\frac{\partial}{\partial m_\mu^2}\right]G^{(n)} = 0}$$

full Callan-Symanzik equation. (1970).

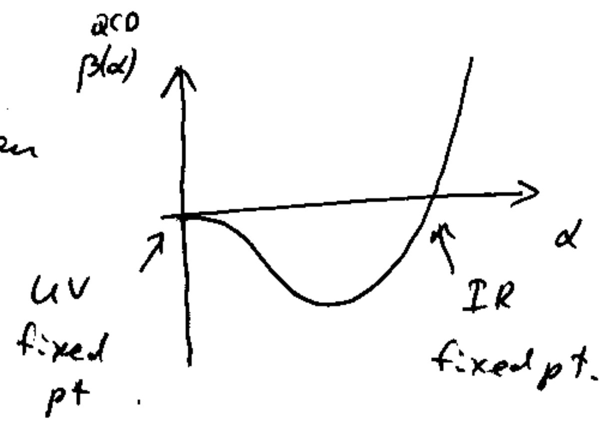
$\Rightarrow$ gives renormalization group (RG) flow of $G^{(n)}$.

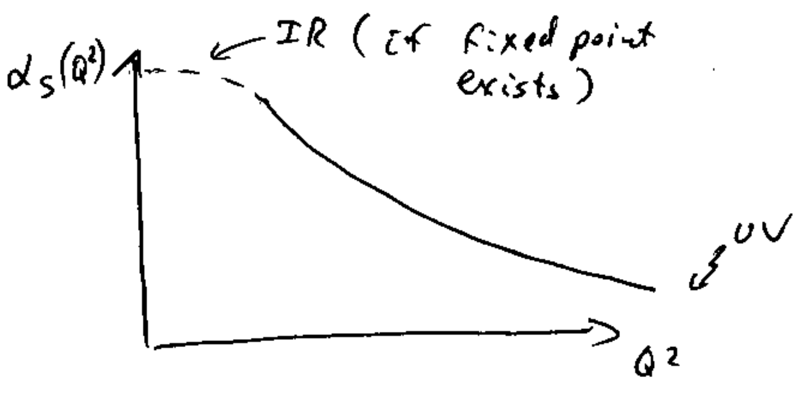
---

other beta-functions: in QCD $\beta(\alpha) < 0$ negative!



may even be $\Rightarrow$





<u>Example</u> Two-point function $G^{(2)}(p)$

$$G^{(2)}(x_1,x_2) = \langle \Psi_0 | T\,\varphi(x_1)\varphi(x_2)|\Psi_0\rangle \Rightarrow G^{(2)}(p) = \frac{i}{p^2}\cdot f\left(p^2/\mu^2\right)$$

in general $\Rightarrow$ Callan-Symanzik eqn becomes:

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu} - 2\gamma(\alpha_\mu)\right] G^{(2)}(p) = 0$$

$\Rightarrow$ replace $\mu^2 \frac{\partial}{\partial \mu^2} \to -p^2\frac{\partial}{\partial p^2} - 1 \quad \Rightarrow$

$$\left[p^2 \frac{\partial}{\partial p^2} - \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu} + 1 + 2\gamma(\alpha_\mu)\right] G^{(2)}(p) = 0$$

The solution is

$$G^{(2)}(p) = \frac{G_0^{(2)}(\alpha(p^2))}{\mu^2}\, e^{-\int_{\mu^2}^{p^2} \frac{dp'^2}{p'^2}\left[1+2\gamma(\alpha(p'^2))\right]}$$

(check!)

where $\alpha(p^2)$ is the physical coupling such that

$$\left[p^2\frac{\partial}{\partial p^2} - \beta(\alpha_\mu)\frac{\partial}{\partial \alpha_\mu}\right]\alpha(p^2) = 0 \quad \text{and} \quad \alpha(\mu^2) = \alpha_\mu .$$

If, for simplicity, we neglect running of the coupling

$\Rightarrow$ & take $\gamma(\alpha) \simeq \frac{1}{2} C\alpha^2 \Rightarrow G^{(2)}(p) = \frac{G_0^{(2)}}{\mu^2}\cdot\left(\frac{p^2}{\mu^2}\right)^{-[1+C\alpha^2]}$

$$\Rightarrow G^{(2)}(p) \propto \left(p^2\right)^{\overbrace{-1}^{\text{normal dimension}}\, -C\alpha^2}$$

$\leftarrow$ anomalous dimension

$$2\int_{\mu^2}^{p^2}\frac{dp'^2}{p'^2}\,\cancel{\mu}\,\underbrace{\beta(\alpha_\mu)\frac{\partial}{\partial\alpha_\mu}}_{=p'^2\frac{\partial}{\partial p'^2}}\gamma(\alpha(p'^2)) = 2\int_{\mu^2}^{p^2}dp'^2\frac{\partial}{\partial p'^2}\gamma(\alpha(p'^2))$$

$$= 2\gamma(\alpha(p^2)) - 2\gamma(\alpha(\mu^2))$$

$\Rightarrow$ makes it work out!

---

free theory: $\beta = \gamma = 0 \quad \Rightarrow \quad p^2\frac{\partial}{\partial p^2}G^{(2)} = -G^{(2)}$

$\Rightarrow \; G^{(2)} = \dfrac{i}{p^2}$ as expected.