Last time | # Functional Integral Quantization

(cont'd)

Path Integral Quantum Mechanics (cont'd)

Non-relativistic 1-particle QM: $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$

$[\hat{q}, \hat{p}] = i\hbar$, $\psi(q,t) = {}_S\langle q(t) | \psi(t) \rangle_S$ (wave function)

$$\rightarrow \psi(q,t) = \int_{-\infty}^{\infty} dq' \, \langle q(t) | e^{-\frac{i}{\hbar}\hat{H}(t-t')} | q'(t') \rangle \, \psi(q',t')$$

Def. Time-evolution (Feynman) kernel:

$$U(q,t;q',t') \equiv {}_S\langle q(t) | e^{-\frac{i}{\hbar}\hat{H}(t-t')} | q'(t') \rangle_S .$$

We showed that

$$U(q,t;q',t') = \lim_{N\to\infty} \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N-1} \frac{dq_i \, dp_i}{2\pi\hbar} \right] \frac{dp_N}{2\pi\hbar} \, e^{\frac{i}{\hbar}\delta t \sum_{j=1}^{N} \left[p_j \frac{q_j - q_{j-1}}{\delta t} - H \right]}$$

& denoted this object by

$$U(q,t;q',t') = \int [\mathcal{D}q \, \mathcal{D}p] \, e^{\frac{i}{\hbar}\int_{t'}^{t} dt'' [p(t'')\dot{q}(t'') - H(p(t''),q(t''))]}$$

path integral.

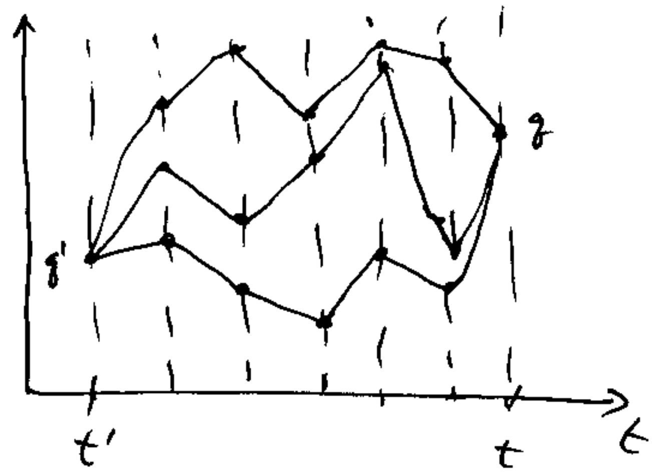
Integrating out $\mathcal{D}p$ we got

$$U(q,t;q',t') = \lim_{N\to\infty} \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N-1} dq_i \right] \left[\frac{m}{2\pi\hbar i \delta t} \right]^{N/2} e^{\frac{i}{\hbar}\delta t \sum_{j=1}^{N} L(q_j, \dot{q}_j)}$$

which we denoted by

$$U(q,t;q',t') = \mathcal{N}\int[\mathcal{D}q]\, e^{\frac{i}{\hbar}\int_{t'}^{t} dt''\, L(q(t''),\, \dot{q}(t''))}$$
$$= \mathcal{N}\int[\mathcal{D}q]\, e^{\frac{i}{\hbar} S(q,t;q',t')}$$

integrate over all paths:



<u>Example</u> Free particle, $V(q)=0$

$$U(q,t;q',t') = \lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\,\delta t}\right]^{N/2}\int_{-\infty}^{\infty}\left[\prod_{i=1}^{N-1} dq_i\right] e^{\frac{i}{\hbar}\delta t\cdot\frac{m}{2}\sum_{i=1}^{N}\frac{(q_i-q_{i-1})^2}{\delta t^2}}$$

$\Rightarrow$ integrated out q_i

Example | Free particle: $V(q) = 0$.

$$U(q,t;q',t') = \lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\delta t}\right]^{N/2}\int_{-\infty}^{\infty}\left[\prod_{i=1}^{N-1} dq_i\right] e^{\frac{i}{\hbar}\delta t\cdot\frac{m}{2}\sum_{j=1}^{N}\frac{(q_j-q_{j-1})^2}{\delta t^2}}$$

First integrate over q_1: need to do this integral:

$$I_1 = \int_{-\infty}^{+\infty} dq_1 \sqrt{\frac{m}{2\pi i\hbar\delta t}}\cdot e^{\frac{i}{2\hbar}\frac{m}{\delta t}\left[(q_1-q_0)^2+(q_2-q_1)^2\right]}.$$

$$[\dots] = 2q_1^2 - 2q_1(q_0+q_2)+q_0^2+q_2^2 = 2\left(q_1-\frac{q_0+q_2}{2}\right)^2 +$$

$$+q_0^2+q_2^2-\frac{(q_0+q_2)^2}{2} = 2\underbrace{\left(q_1-\frac{q_0+q_2}{2}\right)}_{\tilde{q}_1}^2+\frac{1}{2}(q_0-q_2)^2.$$

$$I_1 = \sqrt{\frac{m}{2\pi i\hbar\delta t}}\; e^{\frac{i}{\hbar}\frac{m}{4\delta t}(q_0-q_2)^2}\cdot\int_{-\infty}^{\infty} d\tilde{q}_1\; e^{\frac{i}{2\hbar}\frac{m}{\delta t}\cdot 2\tilde{q}_1^2}$$

$$= \sqrt{\frac{\cancel{m}}{2\cancel{\pi} i\cancel{\hbar}\cancel{\delta t}}}\; e^{\frac{i}{\hbar}\frac{m}{4\delta t}(q_0-q_2)^2}\cdot\sqrt{\frac{\cancel{\pi}\cancel{\hbar}\cancel{\delta t}}{-\cancel{m}i}} = \frac{1}{\sqrt{2}}e^{\frac{i}{\hbar}\frac{m}{2\delta t}(q_0-q_2)^2}$$

$$\Rightarrow U(q,t;q',t') = \sqrt{\frac{m}{2\pi i\hbar\, 2\delta t}}\lim_{N\to\infty}\left[\frac{m}{2\pi i\hbar\delta t}\right]^{\frac{N-2}{2}}\int_{-\infty}^{\infty}\left[\prod_{i=2}^{N-1}dq_i\right]\cdot$$

$$\cdot\, e^{\frac{i}{\hbar}\delta t\frac{m}{2}\left[\frac{1}{2}\left(\frac{q_0-q_2}{\delta t}\right)^2+\sum_{j=3}^{N}\left(\frac{q_j-q_{j-1}}{\delta t}\right)^2\right]}$$

$\Rightarrow$ note new $1/2$ factors $\Rightarrow$ iterate $\Rightarrow$ in the end one gets

$$U_{\text{free}}(q,t;q',t') = \sqrt{\frac{m}{2\pi i\hbar N\delta t}} \cdot \exp\left\{\frac{i}{\hbar}\frac{1}{N\delta t}\cdot\frac{m}{2}(q-q')^2\right\}$$

$\Rightarrow$ as $N\delta t = t - t'$ $\Rightarrow$

$$\boxed{U_{\text{free}}(q,t;q',t') = \sqrt{\frac{m}{2\pi i\hbar(t-t')}} \cdot \exp\left\{\frac{i}{\hbar}\cdot\frac{m}{2}\cdot\frac{(q-q')^2}{t-t'}\right\}}$$

Feynman kernel for a free particle.

& one can also do the integral for harmonic oscillator, $V(q) = \frac{m}{2}\omega^2 q^2$.

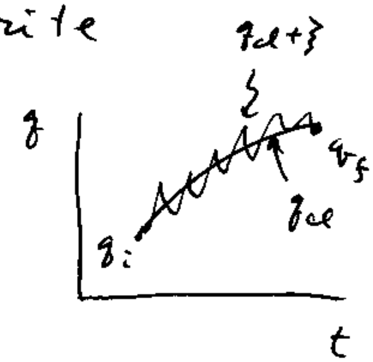
---

as $U(q,t;q',t') = N\int[\mathcal{D}q]\, e^{\frac{i}{\hbar}S(q,t;q',t')}$ $\Rightarrow$

$\Rightarrow$ when $\hbar \to 0$ recover classical physics, have to minimize the action S. $\Rightarrow$ get classical EOM.

Quasi-classical approximation: write

$$q(t) = \underbrace{q_{cl}(t)}_{\text{classical trajectory}} + \underbrace{\xi(t)}_{\text{quantum fluctuations}}$$



Demand that $q_{cl}(t_i) = q_i$, $q_{cl}(t_f) = q_f$

$$\xi(t_i) = \xi(t_f) = 0$$

Expand the Lagrangian

$$L(q,\dot q) = L(q_{cl}+\xi,\ \dot q_{cl}+\dot\xi) = L(q_{cl},\dot q_{cl}) +$$

$$+ \frac{1}{2}\xi^2 \frac{\partial^2 L}{\partial q_{cl}^2} + \frac{1}{2}\dot\xi^2 \frac{\partial^2 L}{\partial \dot q_{cl}^2} + \xi\dot\xi \frac{\partial^2 L}{\partial q_{cl}\partial \dot q_{cl}} + o(\xi^3).$$

(Terms linear in ξ vanish due to classical EOM.)

$\mathcal{D}q \to \mathcal{D}\xi$ as q_{cl} is fixed by initial/final conditions. We get

$$\mathcal{U}(q_f,t_f;q_i,t_i) \underset{\substack{\uparrow \\ \text{quasi-classical}}}{\approx} N e^{\frac{i}{\hbar}S_{cl}} \cdot \int[\mathcal{D}\xi]\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[\frac{1}{2}\xi^2 \frac{\partial^2 L}{\partial q_{cl}^2} + \right.}$$

$$\left. + \frac{1}{2}\dot\xi^2 \frac{\partial^2 L}{\partial \dot q_{cl}^2} + \xi\dot\xi \frac{\partial^2 L}{\partial q_{cl}\partial \dot q_{cl}}\right]$$

$\Rightarrow$ as the integral is quadratic in ξ $\Rightarrow$ can calculate. (see Sterman, pp. 65–66).

$\Rightarrow$ We will not do the general case, but will only work out an example.

Example | Harmonic oscillator:

$$L = \frac{1}{2}m\dot q^2 - \frac{1}{2}m\omega^2 q^2 \Rightarrow m\ddot q = -m\omega^2 q$$

classical EOM

Writing $q(t) = q_{cl}(t) + \xi(t) \Rightarrow$

$$L = \frac{1}{2} m (\dot{q}_{cl} + \dot{\xi})^2 - \frac{1}{2} m\omega^2 (q_{cl} + \xi)^2$$

$$\Rightarrow S[q_{cl} + \xi] = \int_{t_i}^{t_f} dt \Big[\frac{1}{2} m \dot{q}_{cl}^2 - \frac{1}{2} m\omega^2 q_{cl}^2 + \underbrace{m \dot{q}_{cl} \dot{\xi}}_{= \underbrace{m \frac{d}{dt}(\xi \dot{q}_{cl})}_{\text{integrates to } \emptyset} - \underbrace{\cancel{m \xi \ddot{q}_{cl}}}_{(EOM)}} +$$

$$+ \frac{1}{2} m \dot{\xi}^2 - \underbrace{\cancel{m\omega^2 q_{cl} \xi}}_{(EOM)} - \frac{1}{2} m\omega^2 \xi^2 \Big]$$

$$= \int_{t_i}^{t_f} dt \left[L(q_{cl}) + \frac{1}{2} m \left(\dot{\xi}^2 - \omega^2 \xi^2 \right) \right].$$

$$\Rightarrow U_{ho}(q_f, t_f; q_i, t_i) = N e^{\frac{i}{\hbar} S_{cl}} \int [D\xi] \, e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt \, \frac{m}{2} \left(\dot{\xi}^2 - \omega^2 \xi^2 \right)}$$

(no approximation, quasiclassics is exact for h.o.)

Need to do the ξ-integral. As $\xi(t_i) = \xi(t_f) = 0$

$\Rightarrow$ expand in Fourier series:

$$\xi(t) = \sum_{n=1}^{\infty} a_n \sin\left[\frac{\pi n (t - t_i)}{t_f - t_i} \right], \quad T = t_f - t_i$$

$$\int_{t_i}^{t_f} dt \cdot \left(\dot{\xi}^2 - \omega^2 \xi^2 \right) = \sum_{n,m=1}^{\infty} a_n a_m \int_{t_i}^{t_f} dt \Big\{ \frac{\pi n}{T} \cdot \frac{\pi m}{T} \cdot \cos\left[\frac{\pi n (t - t_i)}{T} \right]$$

$$\cdot \cos\left[\frac{\pi m (t - t_i)}{T} \right] - \omega^2 \sin\left[\frac{\pi n (t - t_i)}{T} \right] \sin\left[\frac{\pi m (t - t_i)}{T} \right] \Big\} =$$

$$= \sum_{n=1}^{\infty} a_n^2 \left(\left(\frac{\pi n}{T} \right)^2 - \omega^2 \right) \cdot \frac{1}{2} \cdot T$$

$$[\mathcal{D}\xi] = [\mathcal{D}a_n]\cdot \text{const}$$

↖ not clear, ftn of $T, m, \hbar$ maybe (but not ω)

$$\Rightarrow \int [\mathcal{D}\xi]\, e^{\frac{i}{\hbar}\frac{m}{2}\int_{t_i}^{t_f} dt\, [\dot{\xi}^2 - \omega^2 \xi^2]} = \text{const}\cdot \int [\mathcal{D}a_n]\cdot$$

$$\cdot\, e^{\frac{i}{\hbar}\cdot\frac{m}{2}\sum_{n=1}^{\infty} a_n^2 \frac{T}{2}\left[\left(\frac{\pi n}{T}\right)^2 - \omega^2\right]} = \text{const}\cdot \prod_{n=1}^{\infty}\int_{-\infty}^{\infty} da_n \cdot e^{\frac{i}{\hbar}\frac{mT}{4}\cdot\left[\left(\frac{\pi n}{T}\right)^2 - \omega^2\right]\cdot}$$

$$\cdot a_n^2 = \text{const}\cdot \prod_{n=1}^{\infty} \sqrt{\frac{\pi\, 4\hbar}{-imT\left[\left(\frac{\pi n}{T}\right)^2 - \omega^2\right]}} = \text{const} \prod_{n=1}^{\infty}\left(\frac{4\pi\hbar T}{-im}\right)^{1/2}\cdot$$

$$\cdot \left(\pi^2 n^2 - \omega^2 T^2\right)^{-1/2} = \text{const}'\cdot \prod_{n=1}^{\infty}\left(\pi^2 n^2 - \omega^2 T^2\right)^{-1/2} =$$

$$= \text{const}''\sqrt{\frac{\omega T}{\sin\omega T}}$$ (see page 240′).

$$\Rightarrow U_{ho}(q_f, t_f; q_i, t_i) = \text{const}''\sqrt{\frac{\omega T}{\sin\omega T}}\, \mathcal{N} e^{\frac{i}{\hbar} S_{cl}}.$$

To fix the constant put $\omega = 0 \Rightarrow$ should get free-particle

$U_{free} \Rightarrow \mathcal{N}\cdot\text{const}'' = \sqrt{\frac{m}{2\pi i\hbar T}}$ from comparison.

$$\Rightarrow \boxed{U_{ho}(q_f, t_f; q_i, t_i) = \sqrt{\frac{m}{2\pi i\hbar T}\cdot\frac{\omega T}{\sin\omega T}}\cdot e^{\frac{i}{\hbar}S_{cl}}}$$

~ can easily calculate S_{cl}, as $q_{cl}(t) = A\cos(\omega t + B)$.

$$\prod_{n=1}^{\infty} \left[\pi^2 n^2 - \omega^2 T^2\right]^{-1/2} = \prod_{n=1}^{\infty} \frac{1}{\pi n} \cdot \left[1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right]^{-1/2} =$$

$$= e^{-\sum_{n=1}^{\infty}\left[\ln \pi n + \frac{1}{2} \ln\left[1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right]\right]}$$

$$\sum_{n=1}^{\infty} \ln \pi = \ln \pi \sum_{n=1}^{\infty} 1 = \ln \pi \cdot \underbrace{\zeta(0)}_{= -1/2} \quad , \quad \zeta(a) = \sum_{n=1}^{\infty} \frac{1}{n^a} .$$

Riemann zeta function.

$$= -\frac{1}{2} \ln \pi .$$

$$\sum_{n=1}^{\infty} \ln n = -\left(\frac{d}{da} \sum_{n=1}^{\infty} \frac{1}{n^a}\right)\bigg|_{a=0} = -\zeta'(0) = \frac{1}{2} \ln(2\pi)$$

$$\Rightarrow \prod_{n=1}^{\infty} \left[\pi^2 n^2 - \omega^2 T^2\right]^{-1/2} = e^{-\left(-\cancel{\frac{1}{2}\ln \pi} + \frac{1}{2}\ln(2\cancel{\pi})\right) - \frac{1}{2}\sum_{n=1}^{\infty} \ln\left[1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right]}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{1}{2}\sum_{n=1}^{\infty} \ln\left[1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right]}$$

Using $\sum_{n=1}^{\infty} \ln\left[1 - \frac{x^2}{n^2 \pi^2}\right] = \ln \sin x - \ln x$ ← GR 1.521.2 we get

$$\prod_{n=1}^{\infty} \left[\pi^2 n^2 - \omega^2 T^2\right]^{-1/2} = \frac{1}{\sqrt{2}} \cdot e^{-\frac{1}{2}\left[\ln \sin \omega T - \ln \omega T\right]}$$

$$= \frac{1}{\sqrt{2}} \sqrt{\frac{\omega T}{\sin \omega T}}$$

$$\Rightarrow \boxed{\prod_{n=1}^{\infty} \left[\pi^2 n^2 - \omega^2 T^2\right]^{-1/2} = \frac{1}{\sqrt{2}} \sqrt{\frac{\omega T}{\sin \omega T}}}$$

Time-ordered product: let's see what time-ordered product of operators looks like in terms of path integrals. Need to go to Heisenberg representation:

$$\hat{q}_H(t) = e^{\frac{i}{\hbar}\hat{H}t}\,\hat{q}_S\, e^{-\frac{i}{\hbar}\hat{H}t}$$

$$|q, \underset{\nearrow}{t}\rangle_H = e^{\frac{i}{\hbar}\hat{H}\,t}\,|q(t)\rangle_S$$

The time when it is an eigenstate/eigenfunction of $\hat{q}_H(t)$ operator. (Indeed states in Heisenberg picture do not evolve with time.) $|q,t\rangle_H$ is not an eigenstate of $\hat{q}(t')$ for $t' \neq t$.

Hence $\mathcal{U}(q_f, t_f; q_i, t_i) = {}_S\langle q_f(t_f)|\, e^{-\frac{i}{\hbar}H(t_f - t_i)}\, |q_i(t_i)\rangle_S$

$$= {}_H\langle q_f, t_f \,|\, q_i, t_i\rangle_H$$

$$\Rightarrow \boxed{\mathcal{U}(q_f, t_f; q_i, t_i) = {}_H\langle q_f, t_f | q_i, t_i\rangle_H} \quad \text{(in Heisenberg representation)}$$

If we need to calculate (from now on all is in H.repr.):

$$\langle q_f, t_f|\,\hat{q}(t)\,|q_i, t_i\rangle = \int_{-\infty}^{\infty} dq\, \langle q_f, t_f|\underbrace{\hat{q}(t)\,|q,t\rangle}_{=\,q(t)|q,\,t\rangle}\langle q, t|q_i, t_i\rangle$$

here $t_i < t < t_f$

$$= \mathcal{N}\int [\mathcal{D}q]\; q(t)\; e^{\frac{i}{\hbar} S(q_f, t_f; q_i, t_i)}$$