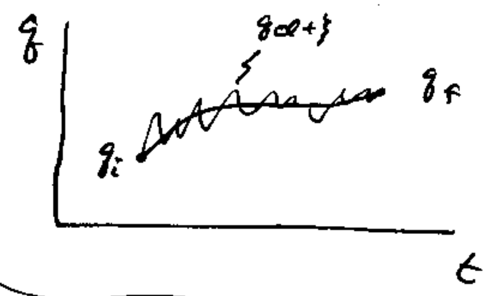
Last time Derived time evolution kernel for a free non-relativistic particle:

$$U_{free}(q,t;q',t') = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}\cdot \exp\left\{\frac{i}{\hbar}\frac{m}{2}\frac{(q-q')^2}{t-t'}\right\}$$

Quasi-classical approximation: $q(t) = q_{cl}(t) + \xi(t)$



$\Rightarrow$ expand the action in ξ

$\Rightarrow$ get

$$U_{qc}(q_f,t_f;q_i,t_i) = N e^{\frac{i}{\hbar}S_{cl}}\cdot\int \mathcal{D}\xi\cdot e^{\frac{i}{\hbar}\int_{t_i}^{t_f}dt\left[\frac{1}{2}\xi^2\frac{\partial^2 L}{\partial q_{cl}^2} + \frac{1}{2}\dot\xi^2\frac{\partial^2 L}{\partial \dot q_{cl}^2} + \xi\dot\xi\frac{\partial^2 L}{\partial q_{cl}\partial \dot q_{cl}}\right]}$$

Example Harmonic oscillator (quasi-classical approximation is exact in this case); we found

$$U_{ho}(q_f,t_f;q_i,t_i) = \sqrt{\frac{m}{2\pi i\hbar T}}\cdot\sqrt{\frac{\omega T}{\sin\omega T}}\; e^{\frac{i}{\hbar}S_{cl}}$$

(The Lagrangian for our harmonic oscillator is

$L = \frac{1}{2}m\dot q^2 - \frac{1}{2}m\omega^2 q^2$.)

$\Rightarrow$ started talking about Time-ordered product:

$$U(q_f,t_f;q_i,t_i) = {}_H\langle q_f,t_f|q_i,t_i\rangle_H.$$ (Heisenberg picture)

Time-ordered product: let's see what time-ordered product of operators looks like in terms of path integrals. Need to go to Heisenberg representation:

$$\hat{q}_H(t) = e^{\frac{i}{\hbar}\hat{H}t}\,\hat{q}_S\,e^{-\frac{i}{\hbar}\hat{H}t}$$

$$|q,t\rangle_H = e^{\frac{i}{\hbar}\hat{H}t}\,|q(t)\rangle_S$$

The time when it is an eigenstate/eigenfunction of $\hat{q}_H(t)$ operator. (Indeed states in Heisenberg picture do not evolve with time.) $|q,t\rangle_H$ is not an eigenstate of $\hat{q}(t')$ for $t'\neq t$.

Hence $\mathcal{U}(q_f,t_f;q_i,t_i) = {}_S\langle q_f(t_f)|\,e^{-\frac{i}{\hbar}H(t_f-t_i)}\,|q_i(t_i)\rangle_S$

$$= {}_H\langle q_f,t_f|q_i,t_i\rangle_H$$

$$\Rightarrow \boxed{\mathcal{U}(q_f,t_f;q_i,t_i) = {}_H\langle q_f,t_f|q_i,t_i\rangle_H}$$ (in Heisenberg representation)

If we need to calculate (from now on all is in H.repr.):

$$\langle q_f,t_f|\,\hat{q}(t)\,|q_i,t_i\rangle = \int_{-\infty}^{\infty} dq\,\langle q_f,t_f|\underbrace{\hat{q}(t)\,|q,t\rangle}_{=\,q(t)|q,t\rangle}\langle q,t|q_i,t_i\rangle$$

here $t_i < t < t_f$

$$= \mathcal{N}\int[\mathcal{D}q]\;q(t)\,e^{\frac{i}{\hbar}S(q_f,t_f;q_i,t_i)}$$

For two operators have

$$\langle q_f, t_f | \hat{q}(t_2)\, \hat{q}(t_1) | q_i, t_i \rangle = \int dq_1\, dq_2 \langle q_f, t_f | \hat{q}(t_2) | q_2, t_2 \rangle$$

$$\langle q_2, t_2 | \hat{q}(t_1) | q_1, t_1 \rangle \langle q_1, t_1 | q_i, t_i \rangle = N \int [\mathcal{D} q]\, q_1(t_1)\, q_2(t_2) \cdot$$

$$\cdot\, e^{\frac{i}{\hbar} S} \qquad \text{if} \quad t_f > t_2 > t_1 > t_i .$$

Not clear what to do if $t_1 > t_2$. However, there's no problem for time-ordered product ($t_f > t_2, t_1 > t_i$):

$$\boxed{\langle q_f, t_f | T\, \hat{q}(t_2)\, \hat{q}(t_1) | q_i, t_i \rangle = N \int [\mathcal{D} q]\, q(t_2)\, q(t_1) \cdot e^{\frac{i}{\hbar} S}}$$

Similarly for N operators: (again $t_f > t_N, \ldots, t_1 > t_i$)

$$\boxed{\langle q_f, t_f | T\, \hat{q}(t_N) \ldots \hat{q}(t_1) | q_i, t_i \rangle = N \int [\mathcal{D} q]\, q(t_N) \ldots q(t_1) \cdot e^{\frac{i}{\hbar} S(q_f, t_f; q_i, t_i)}}$$

<u>Vacuum-to-vacuum transition amplitude</u>

Suppose the system has a unique ground state $|0, T\rangle_H$ $\Rightarrow$ turn on a source $j(t)$ such that

time

$$L \to L + \hbar\, j(t)\, q(t)$$

$\Rightarrow$ ground state is now labeled $|0, T\rangle_H^j$.

We want to find vacuum-to-vacuum transition amplitude in the presence of a source j:

$$\boxed{Z[j] \propto {}^{j}\langle 0, +\infty | 0, -\infty \rangle^{j}}$$

time time

One can show that, according to the general formula,

$$\boxed{Z[j] = \int [\mathcal{D}q]\, e^{\frac{i}{\hbar}\int_{-\infty(1-i\varepsilon)}^{\infty(1-i\varepsilon)} dt\, [L + \hbar\, j(t)\, q(t)]}}$$

$(1-i\varepsilon)$-terms insure that we pick out vacuum states at time $= \pm\infty$ (e.g. Ryder, pp. 180-181).

Consider $\dfrac{\delta Z[j]}{\delta j(t_1)\,\delta j(t_2)}$.

Def. Functional derivative:

$$\boxed{\frac{\delta f(x)}{\delta f(y)} = \delta^{(4)}(x-y)}$$

$$\Rightarrow \frac{\delta}{\delta f(x)} \int d^4y\, f(y)\cdot \varphi(y) = \int d^4y\, \delta^{(4)}(x-y)\, \varphi(y) = \varphi(x).$$

$$\Rightarrow \frac{\delta Z[j]}{\delta j(t_1)\,\delta j(t_2)} = i^2 \int [\mathcal{D}q]\, q(t_1)\, q(t_2)\, e^{\frac{i}{\hbar}\int dt\, [L + \hbar j q]}$$

$\Rightarrow$

$$\langle 0,+\infty| T\, \hat{q}(t_1)\, \hat{q}(t_2) |0,-\infty\rangle = (-i)^2 \mathcal{N} \left.\frac{\delta^2 Z[j]}{\delta j(t_1)\, \delta j(t_2)}\right|_{j=0} .$$

For N-point function get

$$\boxed{\langle 0,+\infty| T\, \hat{q}(t_N) \ldots \hat{q}(t_1) |0,-\infty\rangle = (-i)^N \left.\frac{\delta^N Z[j]}{\delta j(t_1) \ldots \delta j(t_N)}\right|_{j=0} \mathcal{N}}$$

$\Rightarrow$ $Z[j]$ is the generating functional for N-point functions.

$\Rightarrow$ know $Z[j]$ $\Rightarrow$ know all correlators in the theory!

Gell-Mann-Low

as $\langle \psi_0 | T\, \hat{q}(t_N) \ldots \hat{q}(t_1) | \psi_0 \rangle = \dfrac{\langle 0,+\infty| T\, \hat{q}(t_N) \ldots \hat{q}(t_1) |0,-\infty\rangle}{\langle 0,+\infty | 0,-\infty\rangle}$

ground state of $\hat{H}$, $\langle 0, t=0|$

$$\Rightarrow \boxed{\langle \psi_0 | T\, \hat{q}(t_N) \ldots \hat{q}(t_1) | \psi_0 \rangle = (-i)^N \frac{1}{Z[j=0]} \left.\frac{\delta^N Z[j]}{\delta j(t_1) \ldots \delta j(t_N)}\right|_{j=0}}$$

such that

$$\boxed{\langle \psi_0 | T\, \hat{q}(t_N) \ldots \hat{q}(t_1) | \psi_0 \rangle = \frac{\int [\mathcal{D}q]\, q(t_N) \ldots q(t_1)\, e^{\frac{i}{\hbar} S[q]}}{\int [\mathcal{D}q]\, e^{\frac{i}{\hbar} S[q]}}}$$

Functional Quantization of the Scalar Field Theory. (put $\hbar = 1$ again)

Again use the analogy between $q \leftrightarrow \varphi(x)$, $p \leftrightarrow \pi(x)$, $L(q, \dot{q}) \leftrightarrow \mathcal{L}(\varphi, \partial_\mu \varphi)$ => replace

$$\int [\mathcal{D}q] \to \int [\mathcal{D}\varphi]$$

$$S = \int dt\, L(q, \dot{q}) \to S = \int d^4x\, \mathcal{L}(\varphi, \partial_\mu \varphi)$$

=> introduce generating functional by

$$\boxed{Z[j(x)] = \int \mathcal{D}\varphi\, e^{i\int d^4x\, [\mathcal{L} + j(x)\varphi(x)]}}.$$

$$\mathcal{H} = \frac{1}{2}\left(\pi^2 + (\vec{\nabla}\varphi)^2\right) + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 \quad \text{for } \varphi^4 \text{ theory}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4.$$

$${}_H\langle \varphi_f, t_f | \varphi_i, t_i \rangle_H = \int \mathcal{D}\varphi\, \mathcal{D}\pi\, e^{i\int d^4x\, [\pi\dot{\varphi} - \mathcal{H}]} =$$

$$= \int \mathcal{D}\varphi\, e^{i\int d^4x\, \mathcal{L}}$$

as

$$\boxed{\langle \Psi_0 | T\{\varphi_H(x_1) \ldots \varphi_H(x_n)\} | \Psi_0 \rangle = \frac{\int \mathcal{D}\varphi\, \varphi(x_1) \ldots \varphi(x_n)\, e^{i\int d^4x\, \mathcal{L}}}{\int \mathcal{D}\varphi\, e^{i\int d^4x\, \mathcal{L}}}}$$

$$\Rightarrow \boxed{\langle \Psi_0 | T\, \varphi_H(x_1) \ldots \varphi_H(x_n) | \Psi_0 \rangle = (-i)^n \frac{1}{Z[0]} \frac{\delta^n Z[j]}{\delta j(x_1) \ldots \delta j(x_n)}\bigg|_{j=0}}$$

$\Rightarrow$ generating functional generates all possible n-point functions.

Free scalar theory

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \Rightarrow$$

picks out vacua at $t = \pm\infty$

$$Z_0[j] = \int \mathcal{D}\varphi\; e^{i\int d^4x\left[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2 - i\varepsilon}{2}\varphi^2 + j\varphi\right]}$$

$$= (\text{parts}) = \int \mathcal{D}\varphi\; e^{-i\int d^4x\left[\frac{1}{2}\varphi(\partial_\mu\partial^\mu + m^2)\varphi - j\varphi\right]}$$

Gaussian integrals: $\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}$.

$$\int_{-\infty}^{\infty} dx_1 \dots dx_n\; e^{-\frac{a_1}{2}x_1^2 - \dots - \frac{a_n}{2}x_n^2} = \frac{(2\pi)^{n/2}}{\sqrt{a_1 a_2 \dots a_n}}.$$

Define an $n\times n$ matrix $A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ & & \ddots \\ 0 & & a_n \end{pmatrix}$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow \int d^n x\; e^{-\frac{1}{2}x^T A x} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

True for any symmetric matrix A, since can always diagonalize: $\underset{\uparrow \text{diagonal}}{A'} = S A S^{-1}$, such that $\det A' = \cancel{\det S} \cdot \det A \, \cancel{\det S^{-1}} = \det A$.

$y = Sx \Rightarrow d^4y = \underbrace{|\det S|}_{=1} d^4x$ as S is orthogonal (or unitary).

Defining $(dx) = d^4x \, (2\pi)^{-4/2}$ get

$$\int (dx) \, e^{-\frac{1}{2} x^T A x} = \frac{1}{\sqrt{\det A}}$$

Similarly, for functional integrals

$$\int \mathcal{D}\varphi \, e^{-\frac{1}{2}\int d^4x \, \varphi(x) \, \hat{D} \, \varphi(x)} = \frac{1}{\sqrt{\det \hat{D}}}.$$

$\hat{D}$ ← some operator

We see that

$$Z_0[j] = \int \mathcal{D}\varphi \, e^{-i\int d^4x \left[\frac{1}{2}\varphi(\Box + m^2)\varphi - j\varphi\right]}$$

$$\Rightarrow \text{write } \frac{1}{2}\varphi\underbrace{(\Box + m^2)}_{\hat{D}}\varphi - j\varphi = \frac{1}{2}\underbrace{(\varphi - j\hat{D}^{-1})}_{\text{"}\tilde{\varphi}^T\text{"}}\hat{D}\underbrace{(\varphi - \hat{D}^{-1}j)}_{\tilde{\varphi}}$$

$$-\frac{1}{2} j \hat{D}^{-1} j \quad \Rightarrow$$

$$Z_0[j] = \int \mathcal{D}\tilde{\varphi} \, e^{-i\int d^4x\left[\frac{1}{2}\tilde{\varphi}\hat{D}\tilde{\varphi} - \frac{1}{2} j \hat{D}^{-1} j\right]}$$

$$\Rightarrow \boxed{Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}} \, e^{i\int d^4x \, \frac{1}{2} j \hat{D}^{-1} j}}$$

$$\hat{D} = \Box + m^2 \underbrace{- i\varepsilon}_{\text{picks out the right vacuum.}}$$

Explanation: more Gaussian integrals:

$$I \equiv \int (dx)\, e^{-\frac{1}{2} x^T A x + J^T \cdot x}$$

$$J = \begin{pmatrix} J^1 \\ \vdots \\ J^n \end{pmatrix} \sim \text{a "vector"}, \quad J^T \cdot x = x^T J$$

$$\Rightarrow I = \int (dx)\, e^{-\frac{1}{2} \underbrace{(x^T - J^T A^{-1})}_{\tilde{x}^T} A \underbrace{(x - A^{-1} J)}_{\tilde{x}} + \frac{1}{2} J^T A^{-1} J}$$

$$= \frac{1}{\sqrt{\det A}} \cdot e^{\frac{1}{2} J^T A^{-1} J}.$$

To find $\hat{D}^{-1}$ we write the integral differently:

$$Z_0[j] = \int \mathcal{D}\varphi\, e^{-i \int d^4x \left[\frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi - j\varphi\right]} = \Big/ \varphi \to \varphi + \varphi_0$$

such that $(\Box + m^2 - i\varepsilon)\phi_0 = j \quad \Rightarrow$

$$Z_0[j] = \int \mathcal{D}\varphi\, e^{-i \int d^4x \left[\frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi + \frac{1}{2}\varphi_0(\Box + m^2 - i\varepsilon)\varphi\right.}$$

$$\left. + \frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi_0 + \frac{1}{2}\varphi_0(\Box + m^2 - i\varepsilon)\varphi_0 - j\varphi - j\varphi_0\right] = (\text{parts, etc})$$

$$= \int \mathcal{D}\varphi\, e^{-i \int d^4x \left[\frac{1}{2}\varphi \underbrace{(\Box + m^2 - i\varepsilon)}_{\hat{D}}\varphi - \frac{1}{2}\varphi_0 \cdot j\right]} =$$

$$= \frac{1}{\sqrt{\det(i\hat{D})}} \cdot e^{\frac{i}{2}\int d^4x\, j \cdot \varphi_0}.$$

$\Rightarrow (\Box + m^2 - i\varepsilon)\, \varphi_0 = j \quad \Rightarrow$ start by noting

that $\left(\Box_x + m^2 - i\varepsilon\right) D_F(x-y) = -i\, \delta^{(4)}(x-y)$

with $$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{i}{p^2 - m^2 + i\varepsilon}$$

$$\Rightarrow \varphi_0(x) = i \int d^4y\; D_F(x-y)\, j(y) = \hat{D}^{-1} j$$

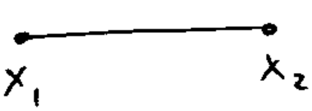
$$\Rightarrow \boxed{Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}} \cdot e^{-\frac{1}{2}\int d^4x\, d^4y\; j(x) D_F(x-y) j(y)}}$$

$\left(\text{btw} \quad \hat{D}^{-1} = i \int d^4y\; D_F(x-y) \right)$.

$$\Rightarrow \langle \Psi_0 | T\, \varphi_H(x_1)\, \varphi_H(x_2) | \Psi_0 \rangle_{free} = (-i)^2 \frac{1}{Z_0(0)} \frac{\delta^2 Z_0[j]}{\delta j(x_1)\, \delta j(x_2)}\bigg|_{j=0}$$

$$= (-)\frac{\delta^2}{\delta j(x_1)\, \delta j(x_2)} \left[e^{-\frac{1}{2}\int d^4x\, d^4y\; j(x) D_F(x-y) j(y)} \right]\bigg|_{j=0}$$

$= D_F(x_1 - x_2) \quad \Rightarrow$ get correct propagator!



(used the fact that $D_F(x-y) = D_F(y-x)$)

$\Rightarrow$ One may also calculate higher order Green

functions: $\langle \Psi_0 | T\, \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) | \Psi_0 \rangle_{free} =$