Last time | Functional Quantization of the Scalar Field Theory (cont'd)

By analogy with QM, we introduced generating functional for Green functions:

$$\boxed{Z[j] = \int \mathcal{D}\varphi \, e^{i\int d^4x \left[\mathcal{L} + j(x)\varphi(x)\right]}}$$

where $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + \mathcal{L}_{int}$.

Any n-point Green function is then

$$\langle\Psi_0|\,T\{\varphi_H(x_1)\ldots\varphi_H(x_n)\}\,|\Psi_0\rangle = \frac{\int\mathcal{D}\varphi\;\varphi(x_1)\ldots\varphi(x_n)\,e^{i\int d^4x\,\mathcal{L}}}{\int\mathcal{D}\varphi\;e^{i\int d^4x\,\mathcal{L}}}$$

or, equivalently,

$$\langle\Psi_0|\,T\{\varphi_H(x_1)\ldots\varphi_H(x_n)\}\,|\Psi_0\rangle = (-i)^n\frac{1}{Z[j=0]}\left.\frac{\delta^n Z[j]}{\delta j(x_1)\ldots\delta j(x_n)}\right|_{j=0}$$

Free Scalar Theory

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 \Rightarrow Z_0[j] = \int\mathcal{D}\varphi\cdot e^{i\int d^4x\left[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2-i\epsilon}{2}\varphi^2 + j\varphi\right]}.$$

We showed that

$$\boxed{\int(dx)\,e^{-\frac{1}{2}x^T A x} = \frac{1}{\sqrt{\det A}}}\,,\quad (dx) = \frac{d^n x}{(2\pi)^{n/2}}\,,\quad A \sim \text{symmetric invertible}$$

$\Rightarrow$ by defining φ_0 using $(\Box + m^2 - i\varepsilon)\varphi_0 = j$ & the above formula we proved that

$$Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}}\, e^{\frac{i}{2}\int d^4x\, j\cdot\varphi_0}$$

where $\hat{D} = \Box + m^2 - i\varepsilon$

$$\Rightarrow \quad \boxed{Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}}\, e^{-\frac{1}{2}\int d^4x\, d^4y\, j(x)\, D_F(x-y)\, j(y)}}$$

where $\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon}}$

(Feynman propagator)

$\Rightarrow (\Box + m^2 - i\varepsilon)\,\varphi_0 = j \quad \Rightarrow$ start by noting

that $\left(\Box_x + m^2 - i\varepsilon\right) D_F(x-y) = -i\,\delta^{(4)}(x-y)$

with $D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{i}{p^2 - m^2 + i\varepsilon}$

$\Rightarrow \varphi_0(x) = i \int d^4y\; D_F(x-y)\, j(y) = \hat{D}^{-1} j$

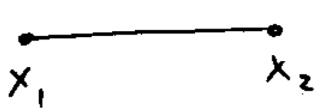
$$\Rightarrow \boxed{Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}} \cdot e^{-\frac{1}{2}\int d^4x\, d^4y\; j(x) D_F(x-y) j(y)}}$$

(btw $\hat{D}^{-1} = i \int d^4y\; D_F(x-y)$).

$$\Rightarrow \langle \Psi_0 | T\,\varphi_H(x_1)\,\varphi_H(x_2) | \Psi_0 \rangle_{free} = (-i)^2 \frac{1}{Z_0(0)} \frac{\delta^2 Z_0[j]}{\delta j(x_1)\,\delta j(x_2)}\Bigg|_{j=0}$$

$$= (-)\frac{\delta^2}{\delta j(x_1)\,\delta j(x_2)} \left[e^{-\frac{1}{2}\int d^4x\, d^4y\; j(x) D_F(x-y) j(y)} \right]\Bigg|_{j=0}$$

$= D_F(x_1 - x_2) \quad \Rightarrow$ get correct propagator!



(used the fact that $D_F(x-y) = D_F(y-x)$)

$\Rightarrow$ One may also calculate higher order Green

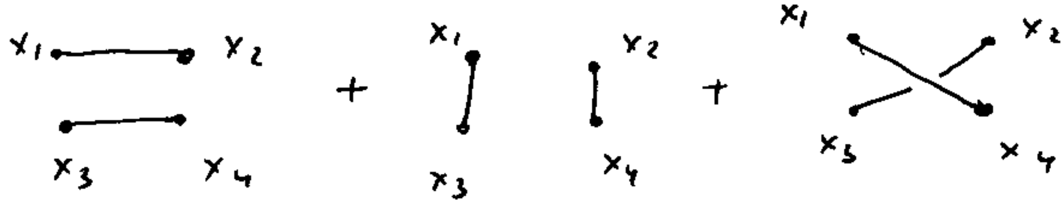
functions: $\langle \Psi_0 | T\,\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) | \Psi_0 \rangle_{free} =$

$$= (-i)^4 \frac{1}{Z_0[0]} \left. \frac{\delta^4 Z_0[j]}{\delta j(x_1)\, \delta j(x_2)\, \delta j(x_3)\, \delta j(x_4)} \right|_{j=0} =$$

$$= \frac{\delta^4}{\delta j(x_1) \ldots \delta j(x_4)} \left. \left\{ e^{-\frac{1}{2} \int d^4x\, d^4y\ j(x)\, D_F(x-y)\, j(y)} \right\} \right|_{j=0}$$

$$= D_F(x_1 - x_2)\, D_F(x_3 - x_4) + D_F(x_1 - x_3)\, D_F(x_2 - x_4) +$$

$$+ D_F(x_1 - x_4)\, D_F(x_2 - x_3).$$



just like before!

φ^4 theory

For the interacting scalar theory with

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi\, \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4!} \varphi^4$$

one has

$$Z[j] = \int \mathcal{D}\varphi \cdot e^{i \int d^4x \left[\frac{1}{2} \partial_\mu \varphi\, \partial^\mu \varphi - \frac{m^2 - i\varepsilon}{2} \varphi^2 - \frac{\lambda}{4!} \varphi^4 + j \cdot \varphi \right]}$$

$\Rightarrow$ this is not a Gaussian integral so it is hard to integrate over φ analytically (try $\int_{-\infty}^{\infty} dx\, e^{-ax^4 - bx^2}$).

Instead we write

$$Z[j] = \quad \cdot e^{i\int d^4x \left(\frac{-\lambda}{4!}\right)\cdot\left(-i\frac{\delta}{\delta j}\right)^4} \cdot \int \mathcal{D}\varphi\, e^{i\int d^4x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2-i\varepsilon}{2}\varphi^2 + j\varphi\right]}$$

$$\Rightarrow \boxed{Z[j] = e^{-i\frac{\lambda}{4!}\int d^4x \frac{\delta^4}{\delta j^4}} Z_0[j]}$$

$\Rightarrow$ Can expand perturbatively in λ $\Rightarrow$ obtain Feynman diagrams and perturbation theory.

Consider 2-point function $\langle \Psi_0 | T\varphi(x_1)\varphi(x_2)|\Psi_0\rangle$:

In general

$$\langle \Psi_0 | T\varphi(x_1)\varphi(x_2)|\Psi_0\rangle = \frac{1}{Z[0]}(-i)^2 \cdot \frac{\delta^2 Z[j]}{\delta j(x_1)\delta j(x_2)}\Bigg|_{j=0}$$

$$= -\frac{\left\{\frac{\delta^2}{\delta j(x_1)\delta j(x_2)} e^{-i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta j^4}} Z_0[j]\right\}\Big|_{j=0}}{\left\{e^{-i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta j^4}} Z_0[j]\right\}\Big|_{j=0}} =$$

$$= -\frac{\left\{\frac{\delta^2}{\delta j(x_1)\delta j(x_2)} e^{-i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta j(x)^4}} \cdot e^{-\frac{1}{2}\int d^4y\, d^4z\, j(y) D_F(y-z) j(z)}\right\}\Big|_{j=0}}{\left\{e^{-i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta j(x)^4}} \cdot e^{-\frac{1}{2}\int d^4y\, d^4z\, j(y) D_F(y-z) j(z)}\right\}\Big|_{j=0}}$$

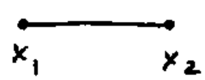
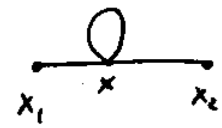
at order-λ^0 just get free theory

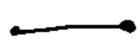
result: $\langle \Psi_0 | T\varphi(x_1)\varphi(x_2)|\Psi_0\rangle\Big|_{O(\lambda^0)} = D_F(x_1 - x_2)$.

at order-λ get:

$$\text{Numerator} = i\frac{\lambda}{4!}\left\{\frac{\delta^2}{\delta j(x_1)\delta j(x_2)}\int d^4x \frac{\delta^4}{\delta j(x)^4}\, e^{-\frac{1}{2}\int d^4y\, d^4z\, j(y)D_F(y-z)j(z)}\right\}\Bigg|_{j=0}$$

$\int d^4x$

$= i\frac{\lambda}{4!}\left\{\frac{1}{3!}\frac{-1}{2^3}\left[3\cdot 2\cdot 4!\right.\right.$   $+\ 3\cdot 2\cdot 2^2\cdot 4!$  $\left.\left.\right]\right\}$

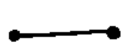
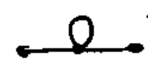
$= -i\lambda\int d^4x\left[\frac{1}{8}\right.$   $+\ \frac{1}{2}$  $\left.\right]$

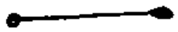
$$\text{DENOMINATOR} = \left[1 - i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta j(x)^4}\right]e^{-\frac{1}{2}\int d^4y\, d^4z\, j(y)D_F(y-z)j(z)}\Bigg|_{j=0}$$

$= 1 - i\frac{\lambda}{\cancel{4!}}\int d^4x\cdot\frac{1}{2!}\left(-\frac{1}{2}\right)^2\cdot\cancel{4!}$ 

$= 1 - i\lambda\int d^4x\ \frac{1}{8}$ 

$\Rightarrow$ get $\langle\Psi_0|T\varphi(x_1)\varphi(x_2)|\Psi_0\rangle =$

$=$ [ $- i\lambda\int d^4x\left[\frac{1}{8}\right.$  $+\ \frac{1}{2}$  $\left.\right] + \dots$] / [$1 - i\lambda\int d^4x\ \frac{1}{8}$  $+ \dots$]

$=$  $- i \lambda \int d^4x \ \frac{1}{2}$  $+ \dots$

$\Rightarrow$ again the denominator cancels all the disconnected graphs!

(Can prove this to all orders similar to the canonical quantization case.)

$\Rightarrow$ We see that we can build the Feynman rules and perturbation theory: they are identical to what we had before.

$\Rightarrow$ For general interaction scalar theory with

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 + \mathcal{L}_{int}(\varphi)$$

write

$$Z[j] = e^{i \int d^4x \, \mathcal{L}_{int}\left(-i \frac{\delta}{\delta j}\right)} Z_0[j]$$

and expand in $\mathcal{L}_{int}$.

n-point functions are given by

$$\langle \psi_0 | T \varphi(x_1) \dots \varphi(x_n) | \psi_0 \rangle = \frac{1}{Z[0]} (-i)^n \frac{\delta^n Z[j]}{\delta j(x_1) \dots \delta j(x_n)} \Big|_{j=0}$$

Finally, let's normalize $Z[j]$ to be 1 at $j=0$, i.e., take $\frac{Z[j]}{Z[0]}$ and write

Def. $$\frac{Z[j]}{Z[0]} = e^{iW[j]}$$

$W[j]$ is the generating functional of connected Green functions.

$$W[j] = -i \ln\{Z[j]/Z[0]\}$$

$$\Rightarrow \frac{\delta W[j]}{\delta j(x_1)\,\delta j(x_2)} = -i \frac{\delta}{\delta j(x_1)}\left[\frac{1}{Z[j]} \cdot \frac{\delta Z[j]}{\delta j(x_2)}\right] =$$

$$= -i \frac{1}{Z[j]} \frac{\delta^2 Z[j]}{\delta j(x_1)\,\delta j(x_2)} + i \frac{1}{Z^2[j]} \frac{\delta Z[j]}{\delta j(x_1)} \frac{\delta Z[j]}{\delta j(x_2)}$$

In φ^4 theory have $\left.\frac{\delta Z}{\delta j}\right|_{j=0} = 0 \Rightarrow$

$$\left.\frac{\delta W[j]}{\delta j(x_1)\,\delta j(x_2)}\right|_{j=0} = -i \frac{1}{Z[0]} \left.\frac{\delta^2 Z[j]}{\delta j(x_1)\,\delta j(x_2)}\right|_{j=0} = i\, D_F(x_1 - x_2) + \dots$$

$= i\langle \Psi_0 | T\,\varphi(x_1)\,\varphi(x_2) | \Psi_0\rangle = i\,[$  $+\dots]$

"connected" means no vacuum bubbles here.

$\Rightarrow$ also works for higher order Green functions.