Last time | Functional Quantization of Scalar Field Theory.

Free field theory: $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2$

$$\Rightarrow Z_0[j] = \int \mathcal{D}\varphi \, e^{i\int d^4x [\mathcal{L}_0 + j\varphi]}$$

$\Rightarrow$ showed that
$$\boxed{Z_0[j] = \frac{1}{\sqrt{\det(i\hat{D})}} e^{-\frac{i}{2}\int d^4x\, d^4y\, j(x) D_F(x-y) j(y)}}$$

where $\hat{D} = \Box + m^2 - i\varepsilon$, $D_F(x-y) \sim$ Feynman propagator

n-point function is (free theory)

$$\langle 0|T\,\varphi(x_1)\ldots\varphi(x_n) = \frac{(-i)^n}{Z_0[0]} \frac{\delta^n Z[j]}{\delta j(x_1)\ldots\delta j(x_n)}\bigg|_{j=0}$$

$\Rightarrow$ showed that $\langle 0|T\,\varphi(x_1)\,\varphi(x_2)|0\rangle = D_F(x_1 - x_2)$

as expected.

Interacting theory: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}(\varphi)$

$$\Rightarrow \boxed{Z[j] = e^{i\int d^4x\, \mathcal{L}_{int}\left(-i\frac{\delta}{\delta j}\right)} Z_0[j]}$$

In particular, for φ^4 theory $\mathcal{L}_{int} = -\frac{\lambda}{4!}\varphi^4$

$$\Rightarrow Z[j] = e^{-i\frac{\lambda}{4!}\int d^4x \frac{\delta^4}{\delta j(x)^4}} Z_0[j].$$

$$\langle \Psi_0 | T\, \varphi(x_1)\varphi(x_2) | \Psi_0 \rangle = \frac{(-i)^2}{Z[0]} \frac{\delta^2 Z[j]}{\delta j(x_1)\, \delta j(x_2)} =$$ $$- i\lambda \int d^4x\, \frac{1}{2}$$  $$+ \dots$$

$\Rightarrow$ showed that the denominator ($Z[0]$) again, just like in Gell-Mann-Low formula, simply removes vacuum bubbles.

Def. $$\frac{Z[j]}{Z[0]} = e^{iW[j]}$$ $\Rightarrow$ $W[j]$ is the generating functional of connected (no vacuum bubbles) Green functions:

$$\frac{\delta W[j]}{\delta j(x_1)\, \delta j(x_2)} = i \Big[$$ $$+$$  $$+ \dots \Big].$$

Functional Quantization of Spinor Fields

In canonical quantization we had anti-commutation relations: $\{\psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)\}=0, \dots$

$\Rightarrow$ need anti-commuting numbers to describe these spinor fields in a functional integral.

Def. Grassmann numbers (generator of Grassmann algebra): θ and η are Grassmann #'s if they anti-commute:

$$\boxed{\theta \cdot \eta = -\eta \cdot \theta}$$

$\Rightarrow \quad \theta^2 = 0, \quad \eta^2 = 0.$

Any function of a Grassman # is

$$f(\theta) = A + B\,\theta$$

Taylor series

A, B ~ regular #'s; $f(\theta) = c_1 + c_2\theta + \underbrace{c_3\theta^2}_{0} + \underbrace{\dots}_{0}$

$$\int d\theta\, f(\theta) = \int d\theta\,[A + B\theta] = \Big|\, \theta \to \theta + \eta \,\Big| =$$

$$= \int d\theta\,[(A + B\eta) + B\theta] \Rightarrow \boxed{\begin{aligned} \int d\theta &= 0 \\ \int d\theta \cdot \theta &= 1 \end{aligned}}$$

~Integrals

~ convention

$$\int d\theta \int d\eta \; \eta \cdot \theta = +1$$

$\uparrow$ convention

$$\frac{\partial}{\partial\theta} f(\theta) = \frac{\partial}{\partial\theta}(A + B\theta) = B \Rightarrow \boxed{\frac{\partial f(\theta)}{\partial\theta} = B}$$

$A, B \sim$ regular complex #'s

$\Rightarrow$ differentiation gives the same result as integration.

$\Rightarrow$ can generalize this to complex Grassman numbers: $\eta = \frac{\eta_1 + i\eta_2}{\sqrt{2}}$, $\bar{\eta} = \frac{\eta_1 - i\eta_2}{\sqrt{2}}$

$\eta_1, \eta_2 \sim$ real Grassmann #'s; $\overline{(\theta\eta)} = \bar{\eta}\,\bar{\theta} = -\bar{\theta}\,\bar{\eta}$.

$$f(\eta, \bar{\eta}) = c_0 + c_1\,\eta + c_2\,\bar{\eta} + c_3\,\bar{\eta}\,\eta$$

$\uparrow$ any function

$$\eta^2 = \frac{1}{2}\cdot\left[\eta_1 + i\eta_2\right]^2 = \frac{1}{2}\Big(\underset{=0}{\eta_1^2} + i\underbrace{(\eta_1\eta_2 + \eta_2\eta_1)}_{=0} + i^2\underset{=0}{\eta_2^2}\Big) = 0$$

$$\bar{\eta}^2 = 0$$

$$\bar{\eta}\,\eta = \frac{1}{2}(\eta_1 - i\eta_2)(\eta_1 + i\eta_2) = \frac{1}{2}\cdot(-i)(\eta_2\eta_1 - \eta_1\eta_2) \neq 0.$$

If θ, η are complex Grassmann #'s $\Rightarrow$

$$(\theta\,\eta)^* = \bar{\eta}\;\bar{\theta} \qquad \text{(like Hermitean conjugation)}$$

$\eta, \bar{\eta}$ ~ independent Grassmann #'s

$$\Rightarrow \boxed{\int d\eta = 0, \quad \int d\eta \cdot \eta = 1, \quad \int d\bar{\eta} = 0, \quad \int d\bar{\eta} \cdot \bar{\eta} = 1}$$

Derivatives:

$$\boxed{\begin{gathered} \frac{\partial}{\partial \eta} \eta = 1, \quad \frac{\partial}{\partial \eta} \bar{\eta} = 0, \ldots \\ \frac{\partial}{\partial \eta} (\eta \bar{\eta}) = \bar{\eta}, \quad \frac{\partial}{\partial \bar{\eta}} (\eta \bar{\eta}) = -\eta \end{gathered}}$$

(left derivative)

$\Rightarrow$ e.g. $\frac{\partial f(\eta, \bar{\eta})}{\partial \eta} = -c_1 - c_3 \bar{\eta}$

if c_1 also Grassmann #

<u>Example</u>

Taylor expansion

$$\int d\bar{\eta}\, d\eta\, e^{-A\bar{\eta}\eta} = \int d\bar{\eta}\, d\eta \left(1 - A\bar{\eta}\eta\right) = -A \cdot$$

$$\int d\bar{\eta} \int d\eta\, \bar{\eta}\eta = A \int d\bar{\eta} \underbrace{\int d\eta \cdot \eta}_{=1} \bar{\eta} = A$$

$$\Rightarrow \boxed{\int d\bar{\eta}\, d\eta\, e^{-A\bar{\eta}\eta} = A}.$$

$\Rightarrow$ can define delta-function $\delta(\eta - \eta_0) = \eta - \eta_0$

check: $\int d\eta\, \delta(\eta - \eta_0) f(\eta) = \int d\eta \cdot (\eta - \eta_0)(A + B\eta) =$

$= A + B\eta_0 = f(\eta_0)$ as desired!

$\Rightarrow$ one may define N-component Grassmann variables:

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}, \quad \bar\eta = (\bar\eta_1, \bar\eta_2, \ldots, \bar\eta_N)$$

$\Rightarrow \bar\eta\eta = \bar\eta_1\eta_1 + \bar\eta_2\eta_2 + \ldots + \bar\eta_N\eta_N$. If A is a Hermitean $N\times N$ matrix $\Rightarrow$

$$\int d^N\bar\eta\, d^N\eta\, e^{-\bar\eta A \eta} = \left|\begin{array}{l} A = U^+ A_D U, \quad A_D \text{ – diagonal} \\ \left.\begin{array}{l} \eta' = U\eta \\ \bar\eta \bar{U} = \bar\eta' \end{array}\right\} \Rightarrow d^N\bar\eta' d^N\eta' = |\det U \cdot \det U^+| \cdot d^N\bar\eta \cdot d^N\eta = d^N\bar\eta\, d^N\eta \end{array}\right.$$

$$= \int d^N\bar\eta'\, d^N\eta'\, e^{-\bar\eta' A_D \eta'} = \int d\bar\eta'_N \ldots d\bar\eta'_1\, d\eta'_1 \ldots d\eta'_N \cdot$$

$$\cdot\, e^{-\bar\eta'_1 a_{11}\eta'_1 - \ldots - \bar\eta'_N a_{NN}\eta'_N} = \det A_D = \det A$$

$$\Rightarrow \boxed{\int d^N\bar\eta\, d^N\eta\, e^{-\bar\eta A \eta} = \det A}$$

$$\int d^N\bar\eta\, d^N\eta\, e^{-\bar\eta A\eta + \bar\theta\eta + \bar\eta\theta} = \int d^N\bar\eta\, d^N\eta\, e^{-\underbrace{(\bar\eta - \bar\theta A^{-1})}_{\bar\xi} \cdot A \cdot \underbrace{(\eta - A^{-1}\theta)}_{\xi} + \bar\theta A^{-1}\theta} = \det A \cdot e^{\bar\theta A^{-1}\theta} \Rightarrow$$

$$\boxed{\int d^N\bar\eta\, d^N\eta\, e^{-\bar\eta A\eta + \bar\theta\eta + \bar\eta\theta} = (\det A)\cdot e^{\bar\theta A^{-1}\theta}}$$

Similarly one can define infinite-dimensional Grassmann numbers $\eta(x)$:

$$\{\eta(x), \eta(y)\} = 0$$

$$\frac{\partial \eta(x)}{\partial \eta(y)} = \delta(x-y)$$

$$\int d\eta(x) = 0, \quad \int d\eta(x) \cdot \eta(x) = 1.$$

$$\boxed{\int \mathcal{D}\bar{\eta}(x) \cdot \mathcal{D}\eta(x)\; e^{-\int \bar{\eta} A \eta d^4x} = \det A}$$

by analogy.

Consider free Dirac field:

$$\mathcal{L} = \bar{\psi}\left[i\not{\partial} - m\right]\psi.$$

$$\Rightarrow \boxed{Z_0[\eta,\bar{\eta}] = \int [\mathcal{D}\bar{\psi}\,\mathcal{D}\psi]\, e^{i\int d^4x\left[\bar{\psi}(i\not{\partial}-m+i\varepsilon)\psi + \bar{\eta}\psi + \bar{\psi}\eta\right]}}$$

is the generating functional for Dirac field Green functions.

Denote $S = i\not{\partial} - m + i\varepsilon \Rightarrow$

$$Z_0[\eta,\bar{\eta}] = \int [\mathcal{D}\bar{\psi}\,\mathcal{D}\psi]\, e^{i\int d^4x\left[\bar{\psi}\,S\,\psi + \bar{\eta}\psi + \bar{\psi}\eta\right]}$$

$$= \int [\mathcal{D}\bar{\psi}\,\mathcal{D}\psi]\, e^{i\int d^4x\left[(\bar{\psi}+\bar{\eta}S^{-1})\,S\,(\psi + S^{-1}\eta) - \bar{\eta}S^{-1}\eta\right]}$$

$$= \det[iS]\, e^{-i\int d^4x\, \bar{\eta}\, S^{-1}\eta}$$

If $S\cdot\psi_0 = \eta \Rightarrow \psi_0 = S^{-1}\eta.$

$$\left[i\not{\partial} - m + i\varepsilon\right]\psi_0 = \eta \Rightarrow \psi_0(x) = -i\int d^4y\, S_F(x-y)\,\eta(y)$$

with $S_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}$ a Dirac propagator.

$$\Rightarrow \boxed{Z_0[\eta,\bar{\eta}] = \det\left(i\left[i\not{\partial} - m\right]\right)\cdot e^{-\int d^4x\, d^4y\, \bar{\eta}(x)\, S_F(x-y)\,\eta(y)}}$$