Last time: Functional Quantization of Spinor Fields

(cont'd)

Def. Grassmann #'s: $\boxed{\theta\eta = -\eta\theta}$ $\Rightarrow \theta^2 = \eta^2 = 0$

$f(\theta) = A + B\theta$ ~ any ftn; $\boxed{\int d\theta = 0, \ \int d\theta \cdot \theta = 1}$

$\boxed{\dfrac{\partial f(\theta)}{\partial\theta} = B}$ as $\dfrac{\partial A}{\partial\theta} = 0$, $\dfrac{\partial B\theta}{\partial\theta} = B$ (left derivative)

$$\underbrace{\eta = \frac{\eta_1 + i\eta_2}{\sqrt{2}} \ ; \ \bar{\eta} = \frac{\eta_1 - i\eta_2}{\sqrt{2}}}_{\text{complex G.\#'s}} \ , \quad \eta_1, \eta_2 \sim \text{real G.\#'s}$$

$$\boxed{\int d\eta = 0, \ \int d\eta \cdot \eta = 1, \ \int d\bar{\eta} = 0, \ \int d\bar{\eta} \cdot \bar{\eta} = 1}$$

$f(\eta, \bar{\eta}) = c_0 + c_1\eta + c_2\bar{\eta} + c_3\bar{\eta}\eta$ ~ general function

$$\left\{\frac{\partial}{\partial\eta}, \eta\right\} = 1, \ \left\{\frac{\partial}{\partial\eta}, \bar{\eta}\right\} = 0, \quad \left\{\frac{\partial}{\partial\bar{\eta}}, \bar{\eta}\right\} = 1, \ \left\{\frac{\partial}{\partial\bar{\eta}}, \eta\right\} = 0$$

left derivative

$$\int d\bar{\eta}\, d\eta\, e^{-A\bar{\eta}\eta} = A$$ useful integral

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}, \quad \bar{\eta} = (\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_N)$$

$$d^N\bar\eta \, d^N\eta = d\bar\eta_N \dots d\bar\eta_1 d\eta_1 \dots d\eta_N = d\bar\eta_N d\eta_N \dots d\bar\eta_1 d\eta_1$$

$\Rightarrow$ we showed that

$$\boxed{\int d^N\bar\eta \, d^N\eta \; e^{-\bar\eta A \eta + \bar\theta \eta + \bar\eta \theta} = (\det A)\, e^{\bar\theta A^{-1} \theta}}$$

$A \sim$ Hermitean $N\times N$ matrix

$\eta(x) \sim$ infinite-dim G. #. $\sim$ Grassmann field

$$\{\eta(x), \eta(y)\} = 0, \quad \frac{\delta \eta(x)}{\delta \eta(y)} = \delta^{(4)}(x-y)$$

$$\int d\eta(x) = 0, \quad \int d\eta(x) \cdot \eta(x) = 1$$

$$\Rightarrow \boxed{\int \mathcal{D}\bar\eta(x)\, \mathcal{D}\eta(x)\; e^{-\int d^4x \, \bar\eta(x) A \eta(x)} = \det A}$$

Consider free Dirac field:

$$\mathcal{L} = \bar{\psi}\,[\,i\not{\partial} - m\,]\,\psi .$$

$$\Rightarrow \boxed{Z_0[\eta,\bar{\eta}] = \int [\mathcal{D}\bar{\psi}\,\mathcal{D}\psi]\, e^{\,i\int d^4x\,[\,\bar{\psi}(i\not{\partial}-m+i\varepsilon)\psi + \bar{\eta}\psi + \bar{\psi}\eta\,]}}$$

is the generating functional for Dirac field Green functions.

Denote $\$ = i\not{\partial} - m + i\varepsilon \Rightarrow$

$$Z_0[\eta,\bar{\eta}] = \int [\mathcal{D}\bar{\psi}\,\mathcal{D}\psi]\, e^{\,i\int d^4x\,[\,\bar{\psi}\,\$\,\psi + \bar{\eta}\psi + \bar{\psi}\eta\,]}$$

$$= \int [\mathcal{D}\bar{\psi}\,\mathcal{D}\psi]\, e^{\,i\int d^4x\,[\,(\bar{\psi} + \bar{\eta}\,\$^{-1})\,\$\,(\psi + \$^{-1}\eta) - \bar{\eta}\,\$^{-1}\eta\,]}$$

$$= \det[-i\,\$]\; e^{-i\int d^4x\,\bar{\eta}\,\$^{-1}\eta}$$

If $\$\cdot\psi_0 = \eta \Rightarrow \psi_0 = \$^{-1}\eta$.

$$[\,i\not{\partial} - m + i\varepsilon\,]\,\psi_0 = \eta \Rightarrow \psi_0(x) = -i\int d^4y\; S_F(x-y)\,\eta(y)$$

with $S_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}$ a Dirac propagator.

$\left(\,[\,i\not{\partial} - m + i\varepsilon\,]\,S_F(x-y) = i\,\delta^{(4)}(x-y)\,\right)$

$$\Rightarrow \boxed{Z_0[\eta,\bar{\eta}] = \det\!\left(-i\,[\,i\not{\partial} - m\,]\right)\cdot e^{-\int d^4x\,d^4y\;\bar{\eta}(x)\,S_F(x-y)\,\eta(y)}}$$

$$\langle \Psi_0 | T\, \psi_\alpha(x_1)\, \bar{\psi}_\beta(x_2) | \Psi_0 \rangle \;\propto\; \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\; \psi_\alpha(x_1)\, \bar{\psi}_\beta(x_2) \cdot$$

$$\cdot\, e^{i\int d^4x\, \bar{\psi}(i\not{\partial}-m)\psi} = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \left\{ \left(-i\frac{\delta}{\delta\bar{\eta}_\alpha(x_1)}\right)\left(i\frac{\delta}{\delta\eta_\beta(x_2)}\right) \cdot \right.$$

$$\left. \cdot\, e^{i\int d^4x\,[\bar{\psi}(i\not{\partial}-m)\psi + \bar{\eta}\psi + \bar{\psi}\eta]} \right\}\Bigg|_{\eta=\bar{\eta}=0}$$

$$\Rightarrow \langle \Psi_0 | T\, \psi_\alpha(x_1)\, \bar{\psi}_\beta(x_2) | \Psi_0 \rangle = \frac{1}{Z_0[0,0]}\; \frac{\delta^2 Z_0[\eta,\bar{\eta}]}{\delta\bar{\eta}_\alpha(x_1)\,\delta\eta_\beta(x_2)}\Bigg|_{\eta=\bar{\eta}=0}$$

$\Rightarrow$ get
$$\langle \Psi_0 | T\, \psi_\alpha(x_1)\, \bar{\psi}_\beta(x_2) | \Psi_0 \rangle = \left\{ \frac{\delta^2}{\delta\bar{\eta}_\alpha(x_1)\,\delta\eta_\beta(x_2)} \cdot \right.$$

$$\left. \cdot\, e^{-\int d^4y\, d^4z\, \bar{\eta}(y)\, S_F(y-z)\, \eta(z)} \right\}\Bigg|_{\eta=\bar{\eta}=0} = S_F(x_1-x_2)$$

$\Rightarrow$ we recover the propagator for Dirac fields

$\Rightarrow$ $Z_0[\eta,\bar{\eta}]$ is indeed a generating functional for Dirac Green functions. (free Dirac theory).

$\Rightarrow$ higher-order correlators are constructed similarly:

$$\psi_\alpha(x_i) \to -i\frac{\delta}{\delta\bar{\eta}_\alpha(x_i)}, \quad \bar{\psi}_\beta(x_j) \to i\frac{\delta}{\delta\eta_\beta(x_j)}.$$

One can construct an interacting theory of scalars and spinors: Yukawa theory.

$$\mathcal{L}_{\text{Yukawa}} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + \bar{\psi}\left[i\partial\!\!\!/ - M\right]\psi - g\,\varphi\bar{\psi}\psi$$

$\Rightarrow$ the generating functional would be

$$\boxed{\begin{aligned} Z_{\text{Yukawa}}[\eta,\bar{\eta},j] &= \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}\varphi\; e^{i\int d^4x\left[\mathcal{L}_{\text{Yukawa}} + \bar{\eta}\psi + \bar{\psi}\eta + \varphi j\right]} \\ &= e^{-i\int d^4x\, g\,(-i)\frac{\delta}{\delta j(x)}\, i\frac{\delta}{\delta\eta(x)}\,(-i)\frac{\delta}{\delta\bar{\eta}(x)}}\; \underbrace{Z_0[\eta,\bar{\eta}]}_{\text{free Dirac theory}}\; \underbrace{Z_0[j]}_{\text{free scalar theory.}} \end{aligned}}$$

$\Rightarrow$ again can construct perturbation theory by expanding in powers of g.

Functional Quantization of Electromagnetic Field

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 \Rightarrow$ Naturally expect

$$Z[j^\mu(x)] = \int \mathcal{D}A_\mu \, e^{i\int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2 + j^\mu(x) A_\mu(x)\right]}$$

Write $\int d^4x \left(-\frac{1}{4}\right) F_{\mu\nu}^2 = (\text{parts}) = \frac{1}{2}\int d^4x \, A^\mu [g_{\mu\nu}\Box - \partial_\mu\partial_\nu] A^\nu$

$$\Rightarrow Z[j^\mu] = \int \mathcal{D}A_\mu \, e^{i\int d^4x \left[\frac{1}{2} A_\mu \underbrace{[g^{\mu\nu}\Box - \partial^\mu\partial^\nu]}_{\hat{D}^{-1\,\mu\nu}} A_\nu + j^\mu A_\mu\right]}$$

$$\propto e^{-\frac{i}{2}\int d^4x \, j^\mu(x)\hat{D}_{\mu\nu} j^\nu(x)}$$

$\Rightarrow$ the photon propagator would be $\dfrac{\delta^2 Z[j]}{\delta j^\mu(x)\delta j^\nu(y)} = i\hat{D}_{\mu\nu}(x-y)$

$\Rightarrow$ need to find $\hat{D}$:

$$[g^{\mu\nu}\Box - \partial^\mu\partial^\nu] \cdot D_{\nu\rho}(x-y) = i\delta^\mu{}_\rho \, \delta^{(4)}(x-y)$$

$\Rightarrow$ hit both sides with $\partial_\mu \Rightarrow$ get

$$0 = i\,\partial_\rho \delta^{(4)}(x-y) \Rightarrow \text{no such } D_{\nu\rho} \text{ exists!}$$

$\Rightarrow$ operator $[g^{\mu\nu}\Box - \partial^\mu\partial^\nu]$ has no inverse!

$\Rightarrow$ the problem is due to the fact that we are integrating over all A_μ, but the action is gauge invariant: $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha(x)$

$\Rightarrow$ the integral $\int \mathcal{D}A_\mu \, e^{i\int d^4x \left(\frac{-1}{4}\right) F_{\mu\nu}^2}$ has a sick infinity due to integrals over $\alpha(x)$ ~ configuration related to a given $A_\mu(x)$ through a gauge transformation.

Choose a gauge condition: $\boxed{G(A) = 0}$

e.g. $G(A) = \partial_\mu A^\mu \Rightarrow$ Lorenz gauge

$$1 = \int \mathcal{D}\alpha(x) \cdot \delta\left(G(A^\alpha)\right) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

with $A_\mu^\alpha(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$.

Lorenz gauge: $\partial^\mu A_\mu^\alpha = \partial^\mu A_\mu + \frac{1}{e}\Box\alpha(x) = G(A^\alpha)$

$$\Rightarrow \frac{\delta G(A^\alpha)}{\delta\alpha} = \frac{1}{e}\Box \Rightarrow \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det\left(\frac{\Box}{e}\right)$$

$\Rightarrow$ independent of A_μ. $\Rightarrow$ we will only consider such gauges where $\frac{\delta G}{\delta\alpha}$ is independent of A_μ.

$$\Rightarrow \int \mathcal{D}A_\mu \, e^{iS[A_\mu]} = \int \mathcal{D}A_\mu \, \mathcal{D}\alpha \, e^{iS[A_\mu]} \, \delta(G(A^\alpha)) \cdot$$

$$\cdot \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \cdot \int \mathcal{D}\alpha \cdot \int \mathcal{D}A_\mu \, e^{iS[A_\mu]} \delta(G(A^\alpha))$$

$$= \left| \begin{array}{l} \mathcal{D}A_\mu \to \mathcal{D}A_\mu^\alpha \\ S[A_\mu] = S[A_\mu^\alpha] \\ \text{drop } \alpha \text{ superscript} \end{array} \right. = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A_\mu e^{iS[A]} \delta(G(A))$$

↑ nothing depends on α in the integrand

$\Rightarrow$ choose a class of Lorenz-like gauges:

$$G(A) = \partial_\mu A^\mu - \omega(x)$$

$$\Rightarrow \int \mathcal{D}A_\mu \, e^{iS[A_\mu]} = \det\left(\frac{\Box}{e}\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A_\mu \, e^{iS[A]} \cdot \delta(\partial_\mu A^\mu - \omega(x))$$

right-hand-side is $\omega(x)$ - independent $\Rightarrow$

$\Rightarrow$ multiply by $\quad 1 = N(\xi)\int \mathcal{D}\omega \, e^{-i\int d^4x \, \frac{\omega^2(x)}{2\xi}}$

$\xi \sim$ just a number, $N(\xi) \sim$ normalization factor

$$\Rightarrow \text{get } \int \mathcal{D}A_\mu \, e^{iS[A_\mu]} = \det\left(\frac{\Box}{e}\right) N(\xi)\left(\int \mathcal{D}\alpha\right) \cdot \int \mathcal{D}A_\mu \cdot$$

$$\cdot \, e^{iS[A] - i\int d^4x \, \frac{1}{2\xi}(\partial_\mu A^\mu)^2}$$