Last time | Finished functional quantization of Dirac field:

$$\boxed{Z_0[\eta,\bar\eta] = \int \mathcal{D}\bar\psi\,\mathcal{D}\psi\; e^{i\int d^4x\,[\bar\psi[i\not\partial - m + i\varepsilon]\psi + \bar\eta\psi + \bar\psi\eta]}}$$

↖ generating functional

$$\boxed{Z_0[\eta,\bar\eta] = \det[-i(i\not\partial - m)]\; e^{-\int d^4x\,d^4y\,\bar\eta(x)\,S_F(x-y)\,\eta(y)}}$$

where $S_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{i(\not k + m)}{k^2 - m^2 + i\varepsilon}$ ~ Dirac propagator

Showed that

$$\langle 0|T\,\psi_\alpha(x_1)\bar\psi_\beta(x_2)|0\rangle = \frac{1}{Z_0[0,0]}\; \frac{\delta^2 Z_0[\eta,\bar\eta]}{\delta\bar\eta_\alpha(x_1)\,\delta\eta_\beta(x_2)}\Bigg|_{\eta=\bar\eta=0}$$

$$= S_F(x_1 - x_2)_{\alpha\beta}$$

General rule:

$$\boxed{\psi_\alpha(x) \to -i\frac{\delta}{\delta\bar\eta_\alpha(x)}\;,\quad \bar\psi_\beta(y) \to i\frac{\delta}{\delta\eta_\beta(y)}}$$

$\Rightarrow$ briefly quantized Yukawa theory as well

Functional Quantization of Electromagnetic Field (cont'd)

$$\boxed{Z[j^\mu] \overset{?}{=} \int \mathcal{D}A_\mu\; e^{i\int d^4x\,[-\frac{1}{4}F_{\mu\nu}^2 + j^\mu A_\mu]}}$$

↖ our guess

problem: $\int \mathcal{D}A_\mu$ has ∞ in it due to gauge freedom

$$A_\mu \to A_\mu + \frac{1}{e}\,\partial_\mu \alpha(x)$$

$\Rightarrow$ choose a gauge $G(A) = 0$

$\Rightarrow$ multiply funct. int. by $1 = \int \mathcal{D}\alpha(x)\ \delta\left(G(A^\alpha)\right) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$

$\Rightarrow$ work in, say, Lorenz gauge where $\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$ is

α-independent, $\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det\left(\frac{\Box}{e}\right)$ in Lorenz gauge

$$\Rightarrow \int \mathcal{D}A_\mu\, e^{iS[A_\mu]} = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A_\mu\, e^{iS[A]}\cdot\delta(G(A)).$$

$$\Rightarrow \int \mathcal{D}A_\mu \, e^{iS[A_\mu]} = \int \mathcal{D}A_\mu \, \mathcal{D}\alpha \, e^{iS[A_\mu]} \delta(G(A^\alpha)) \cdot$$

$$\cdot \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \cdot \int \mathcal{D}\alpha \cdot \int \mathcal{D}A_\mu \, e^{iS[A_\mu]} \delta(G(A^\alpha))$$

$$= \left| \begin{array}{l} \mathcal{D}A_\mu \to \mathcal{D}A_\mu^\alpha \\ S[A_\mu] = S[A_\mu^\alpha] \\ \text{drop } \alpha \text{ superscript} \end{array} \right. = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A_\mu e^{iS[A]} \delta(G(A))$$

nothing depends on α in the integrand

$\Rightarrow$ choose a class of Lorenz-like gauges:

$$G(A) = \partial_\mu A^\mu - \omega(x)$$

$$\Rightarrow \int \mathcal{D}A_\mu \, e^{iS[A_\mu]} = \det\left(\frac{\Box}{e}\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A_\mu \, e^{iS[A]} \cdot \delta(\partial_\mu A^\mu - \omega(x))$$

right-hand-side is $\omega(x)$-independent $\Rightarrow$

$\Rightarrow$ multiply by $1 = N(\xi) \int \mathcal{D}\omega \, e^{-i\int d^4x \, \frac{\omega^2(x)}{2\xi}}$

ξ ~ just a number, $N(\xi)$ ~ normalization factor

$$\Rightarrow \text{get } \int \mathcal{D}A_\mu \, e^{iS[A_\mu]} = \det\left(\frac{\Box}{e}\right) N(\xi) \left(\int \mathcal{D}\alpha\right) \cdot \int \mathcal{D}A_\mu \cdot$$

$$\cdot \, e^{iS[A] - i\int d^4x \frac{1}{2\xi}(\partial_\mu A^\mu)^2}$$

$\Rightarrow$ take a gauge-invariant operator $\hat{O}(A)$

$$\Rightarrow \langle \Psi_0 | T\, \hat{O}(A) | \Psi_0 \rangle = \frac{\int \mathcal{D}A_\mu \, O(A)\, e^{iS[A]}}{\int \mathcal{D}A_\mu \, e^{iS[A]}} \Rightarrow$$

plugging in the above expression & cancelling $N(\xi)$, $\det\left(\frac{\Box}{e}\right)$, $\int \mathcal{D}\alpha$ we get

$$\langle \Psi_0 | T\, \hat{O}(A) | \Psi_0 \rangle = \frac{\int \mathcal{D}A_\mu \, O(A)\, e^{i\int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]}}{\int \mathcal{D}A_\mu \, e^{i\int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right]}}$$

$\Rightarrow$ we have a new Lagrangian

$$\mathcal{L}_{new} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

$\Rightarrow$ integrating by parts gives in the exponent

$$e^{i\int d^4x \; \frac{1}{2} A_\mu \left(g^{\mu\nu}\Box - \partial^\mu\partial^\nu\left(1-\frac{1}{\xi}\right)\right) A_\nu}$$

$\Rightarrow$ photon propagator is given by

$$\left[g^{\mu\nu}\Box - \partial^\mu\partial^\nu\left(1-\frac{1}{\xi}\right)\right] D_{\nu\rho}(x-y) = i\delta^\mu{}_\rho \, \delta^{(4)}(x-y)$$

$\Rightarrow$ this operator is invertible $\Rightarrow$ let's go to momentum space

$$\boxed{D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik.(x-y)}\, \tilde{D}_{\mu\nu}(k)}$$

$\Rightarrow \left[-k^2 g^{\mu\nu} + k^\mu k^\nu\left(1-\frac{1}{\xi}\right)\right]\tilde{D}_{\nu\rho}(k) = i\delta^\mu{}_\rho$

$\Rightarrow$ look for $\tilde{D}_{\nu\rho}(k) = A\, g_{\nu\rho} + k_\nu k_\rho B$ with

$A = A(k^2),\ B = B(k^2) \quad \Rightarrow$

$$\left[-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right]\left[A g_{\nu\rho} + B k_\nu k_\rho\right] = i\delta^\mu{}_\rho$$

$$-Ak^2\,\delta^\mu{}_\rho + \left(1-\frac{1}{\xi}\right)A\, k^\mu k_\rho - Bk^2 k^\mu k_\rho + B\left(1-\frac{1}{\xi}\right)k^2 k^\mu k_\rho = i\delta^\mu{}_\rho$$

$\Rightarrow$ equating coefficients of $\delta^\mu{}_\rho$ and $k^\mu k_\rho$ we get

$$-Ak^2 = 1 \Rightarrow A = -\frac{i}{k^2}$$

$$\left(1-\frac{1}{\xi}\right)A - \cancel{Bk^2} + Bk^2\left(\cancel{1}-\frac{1}{\xi}\right) = 0 \Rightarrow B = \frac{\xi-1}{k^2}A$$

$$\Rightarrow B = i\frac{1-\xi}{(k^2)^2} \Rightarrow \tilde{D}_{\mu\nu}(k) = -\frac{i}{k^2}g_{\mu\nu} + ik_\mu k_\nu \frac{1-\xi}{(k^2)^2}$$

$$= \frac{-i}{k^2}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

$\xi = 1$ Feynman gauge
$\xi = 0$ Landau gauge
$\xi = 3$ Yennie gauge

$$\Rightarrow \tilde{D}_{\mu\nu}(k) = \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

as advertised in 1st quarter. ($\xi = \infty \sim$ no gauge fixing $\Rightarrow$ get ∞ propagator.)