Non-Abelian Gauge Theories

~ we will consider theories with $SU(N)$ local gauge symmetry.

~ to construct Lagrangian for such theories start with $U(1)$ symmetry

Abelian Gauge Theories (brief review)

take Dirac field: $\mathcal{L} = \bar{\psi}\left[i\not{\partial} - m\right]\psi$

$\Rightarrow$ $\mathcal{L}$ is invariant under a global $U(1)$ symmetry: $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to \bar{\psi}e^{-i\alpha}$

α ~ real number

$\Rightarrow$ make it local: require that the Lagrangian has local $U(1)$ symmetry:

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{-i\alpha(x)}$$

$$\bar{\psi}\,[i\gamma\cdot\partial - m]\,\psi \to \bar{\psi}\, e^{-i\alpha(x)}\,[i\gamma\cdot\partial - m]\, e^{i\alpha(x)}\psi$$

$$= \bar{\psi}\,[\,i\gamma\cdot\partial + i\gamma\cdot\partial(i\alpha) - m\,]\,\psi = \bar{\psi}\,[i\gamma\cdot\partial - m]\,\psi$$

$$- \bar{\psi}\,\gamma^\mu(\partial_\mu\alpha)\,\psi$$

$\Rightarrow \mathcal{L}$ is not invariant under local $U(1)$ symmetry!

$\Rightarrow$ Fix it by introducing local gauge field $A_\mu(x)$ (<u>gauge</u> the Lagrangian)

$$\boxed{\mathcal{L} = \bar{\psi}\,[\,i\gamma\cdot\partial - m\,]\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - g\,\bar{\psi}\gamma^\mu A_\mu\psi}$$

$\Rightarrow$ require that: $\boxed{\begin{aligned}&\psi \to e^{i\alpha(x)}\psi\\ &A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x)\end{aligned}}$

$$\Rightarrow \mathcal{L} \to \mathcal{L}' = \bar{\psi}\,[\,i\gamma\cdot\partial - m\,]\,\psi - \cancel{\bar{\psi}\,\gamma^\mu(\partial_\mu\alpha)\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$+ e\bar{\psi}\,\gamma^\mu A_\mu\psi + \cancel{\bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi} = \mathcal{L}$$

$\Rightarrow$ now it is invariant!

$\Rightarrow$ Def. Covariant derivative $\boxed{D_\mu \equiv \partial_\mu + ieA_\mu}$

$$\Rightarrow \boxed{\mathcal{L}_{QED} = \bar{\psi}\,[\,i\gamma^\mu D_\mu - m\,]\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

$-\frac{i}{e}[D_\mu, D_\nu]$

as usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F_{\mu\nu} = [D_\mu D_\nu - D_\nu D_\mu]\frac{-i}{e}$

SU(2) Gauge Theory.

Now imagine a theory with a non-abelian symmetry, like $SU(2)$: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $\psi_1, \psi_2 \sim$ spinors

ψ_1 & ψ_2 are different by some quantum #
(e.g. color, weak isospin)

$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m \right]\psi$ with $m = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ is

invariant under $\psi \to \psi' = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\psi$

$\vec{\sigma}$ are Pauli matrices in $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ space.

$\Rightarrow$ <u>global</u> $SU(2)$ symmetry.

$\Rightarrow$ let's make it <u>local</u> (gauge it): $\vec{\alpha} = \vec{\alpha}(x)$

$$\Rightarrow \psi \to \psi' = e^{i\vec{\alpha}(x)\cdot\frac{\vec{\sigma}}{2}}\psi(x) \equiv S(x)\,\psi(x)$$

with $S^+ S = S\, S^+ = \mathbb{1}$.

$$\Rightarrow \mathcal{L} \to \mathcal{L}' = \bar{\psi}\, S^+\left[i\gamma^\mu \partial_\mu - m\right] S\,\psi = \bar{\psi}\left[i\gamma\cdot\partial - m\right]\psi$$

$$+\ \bar{\psi}\, i\gamma^\mu \left(S^+ \partial_\mu S\right)\psi \quad \Rightarrow \text{not invariant}$$

$\Rightarrow$ add a gauge field A_μ^a, $a = 1, 2, 3$:

$$\mathcal{L} = \bar{\psi}\left[i\gamma\cdot\partial - m\right]\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + g\,\bar{\psi}\gamma^\mu A_\mu^a \frac{\sigma^a}{2}\psi$$

$$\mathcal{L} \to \mathcal{L}' = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi + \bar{\psi}\, i\gamma^\mu \left(S^+ \partial_\mu S\right)\psi +$$

$$+ g\bar{\psi}\gamma^\mu S^+ A'_\mu S \psi - \frac{1}{4} F'^a_{\mu\nu} F'^{a\mu\nu}$$

where $A_\mu = A_\mu^a \frac{\sigma^a}{2}$ is a matrix.

Collect ψ-terms: $g\bar{\psi}\gamma^\mu \underbrace{\left[S^+ A'_\mu S + \frac{i}{g} S^+ \partial_\mu S\right]}_{\text{require } = A_\mu}\psi$

$\Rightarrow A_\mu = S^+ A'_\mu S + \frac{i}{g} S^+ \partial_\mu S \Rightarrow S A_\mu S^+ = A'_\mu +$
$+ \frac{i}{g}(\partial_\mu S) S^+ \Rightarrow$

$$A'_\mu = S A_\mu S^+ - \frac{i}{g}(\partial_\mu S) S^+$$
$$\psi' = S\psi$$

non-abelian gauge transformation!

Def. Covariant derivative $D_\mu = \partial_\mu - igA_\mu$

(note: now it's a matrix!)

$$\Rightarrow \mathcal{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

But: we never checked the invariance of $F^a_{\mu\nu} F^{a\mu\nu}$ term. What is $F^a_{\mu\nu}$ anyway? Using abelian analogy write $F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right]$

where $F_{\mu\nu} = F_{\mu\nu}^a \frac{\sigma^a}{2}$.

$$F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right] = \frac{i}{g}\left[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu\right] =$$

$$= \frac{i}{g}\left\{-ig\left[\partial_\mu, A_\nu\right] - ig\left[A_\mu, \partial_\nu\right] - g^2\left[A_\mu, A_\nu\right]\right\}$$

$$= \frac{i}{g}\left\{-ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - g^2\left[A_\mu, A_\nu\right]\right\} =$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$$

$$\Rightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$$

$$F_{\mu\nu}^a \frac{\sigma^a}{2} = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\frac{\sigma^a}{2} - ig A_\mu^b A_\nu^c \underbrace{\left[\frac{\sigma^b}{2}, \frac{\sigma^c}{2}\right]}_{i\varepsilon^{bca}\frac{\sigma^a}{2}} \leftarrow SU(2)$$

$$= \frac{\sigma^a}{2}\left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\varepsilon^{abc} A_\mu^b A_\nu^c\right]$$

$$\Rightarrow \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\,\varepsilon^{abc} A_\mu^b A_\nu^c$$

~ true for $SU(2)$

~ other groups have different group structure constants:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c.$$

What happens to $F_{\mu\nu}$ under non-Abelian gauge transform?

Start with D_μ: $D_\mu = \partial_\mu - ig A_\mu \rightarrow$

$$\rightarrow \partial_\mu - ig\left[S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S) S^{-1}\right] =$$

$$= S\left[\partial_\mu - ig A_\mu\right] S^{-1} = S D_\mu S^{-1}$$

as $S \partial_\mu S^{-1} = \partial_\mu + S(\partial_\mu S^{-1})$

now: $\mathbb{1} = S S^{-1} \Rightarrow 0 = \partial_\mu (S S^{-1}) = (\partial_\mu S) S^{-1} + S(\partial_\mu S^{-1})$

$\Rightarrow S(\partial_\mu S^{-1}) = -(\partial_\mu S) S^{-1} \Rightarrow S \partial_\mu S^{-1} = \partial_\mu - (\partial_\mu S) S^{-1}$

$$\Rightarrow \boxed{D_\mu \rightarrow S D_\mu S^{-1}}$$

$$\Rightarrow F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \rightarrow \frac{i}{g}[S D_\mu S^{-1}, S D_\nu S^{-1}]$$

$$= \frac{i}{g} S [D_\mu, D_\nu] S^{-1} = S F_{\mu\nu} S^{-1}$$

$$\Rightarrow \boxed{F_{\mu\nu} \rightarrow F'_{\mu\nu} = S F_{\mu\nu} S^{-1}}$$

$\Rightarrow$ Note that $F_{\mu\nu}$ is <u>not</u> invariant under gauge transformation if it is non-Abelian!

$$-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} \operatorname{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$$

as $\operatorname{tr}\left(\frac{\sigma^a}{2} \frac{\sigma^b}{2}\right) = \frac{1}{2}\delta^{ab}$ $\Rightarrow$ under non-abelian gauge transformation have

$$-\frac{1}{2} \operatorname{tr}\left(F_{\mu\nu} F^{\mu\nu}\right) \longrightarrow -\frac{1}{2} \operatorname{tr}\left(F'_{\mu\nu} F'^{\mu\nu}\right) = -\frac{1}{2} \operatorname{tr}\left[\cancel{S} F_{\mu\nu} \cancel{S^{-1}} \cancel{S} F^{\mu\nu} \cancel{S^{-1}}\right] = -\frac{1}{2} \operatorname{tr}\left[F_{\mu\nu} F^{\mu\nu}\right]$$

$\Rightarrow$ the Lagrangian is invariant under non-Abelian gauge transformation:

$$\boxed{\mathcal{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{2}\operatorname{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)}$$

true for any gauge group $SU(N)$

$$D_\mu = \partial_\mu - i g A_\mu \quad , \quad F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right]$$