Last time | Non-Abelian Gauge Theories (cont'd)

SU(2) Gauge Theory

Start with $\mathcal{L} = \bar{\psi}[i\not{\partial} - m]\psi$ with $\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$, $m = \begin{pmatrix} m & 0 \\ 0 & m\end{pmatrix}$.

$\mathcal{L}$ has a global $SU(2)$ symmetry. Make it local $\Rightarrow$

$$\boxed{\mathcal{L} = \bar{\psi}[i\not{D} - m]\psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}}$$

with $D_\mu = \partial_\mu - igA_\mu$, $\quad A_\mu = \sum_{a=1}^{3} A_\mu^a \frac{\sigma^a}{2}$,

$\quad$ ← new gauge field

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

$$F_{\mu\nu} = \sum_{a=1}^{3} F^a_{\mu\nu} \frac{\sigma^a}{2} \Rightarrow F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\varepsilon^{abc} A^b_\mu A^c_\nu$$

$\mathcal{L}$ is now invariant under a local $SU(2)$ gauge

symmetry:
$$\begin{cases} A_\mu \to A'_\mu = S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1} \\ \psi \to \psi' = S\psi \\ \bar{\psi} \to \bar{\psi}' = \bar{\psi} S^{-1} \end{cases}$$

with S a 2×2 unitary matrix ($SS^\dagger = S^\dagger S = \mathbb{1}$)

$$D_\mu \to S D_\mu S^{-1} \Rightarrow F_{\mu\nu} \to F'_{\mu\nu} = S F_{\mu\nu} S^{-1}$$

$$\Rightarrow \; -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} \operatorname{tr}\left(F_{\mu\nu} F^{\mu\nu}\right) \rightarrow -\frac{1}{2} \operatorname{tr}\left(S F_{\mu\nu} \cancel{S^{-1}} \cdot \cancel{S} \cdot F^{\mu\nu} S^{-1}\right) = -\frac{1}{2} \operatorname{tr}\left(S F_{\mu\nu} F^{\mu\nu} S^{-1}\right) = -\frac{1}{2} \operatorname{tr}\left(\cancel{S^{-1}} \cancel{S} F_{\mu\nu} F^{\mu\nu}\right)$$

$$= -\frac{1}{2} \operatorname{tr}\left(F_{\mu\nu} F^{\mu\nu}\right) \;\Rightarrow\; \text{invariant!}$$

$$\bar{\psi}\left[i \not{\partial} - m\right]\psi \rightarrow \bar{\psi} S^{-1}\left[i S \not{\partial} S^{-1} - m\right] S \psi =$$

$$= \bar{\psi}\left[i \not{\partial} - m\right]\psi \;\Rightarrow\; \text{invariant!}$$

Generalization: SU(N) Gauge Theory

For a $SU(N)$ gauge theory use $N \times N$ generators of $su(N)$ in the fundamental representation T^a:

$$[T^a, T^b] = i f^{abc} T^c$$

f^{abc} ← structure constants

$\Rightarrow A_\mu = A_\mu^a T^a \quad , \quad a = 1, \ldots, N^2-1$

$F_{\mu\nu} = F_{\mu\nu}^a T^a \Rightarrow$ again $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$ with the covariant derivative $D_\mu = \partial_\mu - ig A_\mu$. One can show that

$$\boxed{F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c}$$

The gauge-invariant Lagrangian is then:

$$\boxed{\mathcal{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}$$

$\psi^i, \; i = 1, \ldots, N \quad \sim N$ different spinors

$A_\mu' = S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S) S^{-1} \; , \; \psi' = S\psi \quad \sim$ gauge transform

$D_\mu \to S D_\mu S^{-1}$

$\Rightarrow \mathcal{L}$ is invariant under $SU(N)$ local gauge symmetry!

Quantum Chromodynamics (QCD): theory of quarks and gluons. SU(3) gauge group

Quark fields: q_α^{if} ← color, $i=1,2,3$; ← flavor index, $f=u,d,s,c,b,t$; ↑ spinor index $\alpha = 1,2,3,4$

A_μ^a ~ gluon fields; a ← color, $a=1,\dots,8$; μ ← Lorentz index $\mu = 0,1,2,3$

The Lagrangian is

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^{if}\left[\, i\gamma\cdot D_{ij} - m_f \right] q^{jf} - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}$$

$D_\mu = \partial_\mu - i g A_\mu^a T^a$, $\quad T^a = \frac{\lambda^a}{2}$ ~ Gell-Mann matrices

⇒ Sum over flavors and colors assumed.

⇒ Other local non-Abelian theories in nature: electroweak interactions (SU(2) group).

Faddeev - Popov Quantization

We want to quantize a gauge theory:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$ (consider a general non-Abelian case).

The generating functional is

$$Z[0] = \int \mathcal{D} A_\mu \, e^{iS} = \int \mathcal{D} A_\mu \, e^{i\int d^4x \left(-\frac{1}{4}\right) F^a_{\mu\nu} F^{a\mu\nu}} =$$

$$= \int \mathcal{D} \bar{A}_\mu \, e^{iS} \cdot \int \mathcal{D} \Lambda$$

where $\bar{A}_\mu$ is the field in one particular gauge, Λ is the gauge transformation.

Problem: $\int \mathcal{D}\Lambda = \infty \Rightarrow Z = \infty \Rightarrow$ bad!

Even worse is the need to pick a gauge: consider abelian field A_μ: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2} A^\mu \underbrace{[g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu]}_{(D^{-1})_{\mu\nu}} A^\nu$

$\Rightarrow$ to find photon propagator need to solve

$$[g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu] D^{\nu\rho}(x) = \delta_\mu^\rho \, \delta^4(x)$$

$\Rightarrow$ act with $\partial^\mu \Rightarrow (\partial \cdot \partial^2 - \partial^2 \partial_\nu) D^{\nu\rho} = 0 = \partial^\rho \delta^4(x)$

$\Rightarrow$ this can not be true $\Rightarrow$ the operator has no inverse! $\Rightarrow$ no photon propagator?

However, if we choose a gauge, e.g. $\partial_\mu A_\mu = 0$

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2} A^\mu \Box A^\nu \Rightarrow g_{\mu\nu} \Box D^{\nu\rho}(x) = \delta_\mu{}^\rho \delta^4(x).$$

$\Rightarrow$ easy to invert!

$\Rightarrow$ Need to fix the gauge!

Start with $\quad Z(0) = \int \mathcal{D} A_\mu \, e^{iS}$.

Insert into the integrand

$$1 = \int \mathcal{D}\Lambda \; \delta(\Lambda) = \int \mathcal{D}\Lambda \cdot \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)$$

where $G(A) = 0$ is the gauge condition we want to impose, e.g. $G(A) = \partial_\mu A^\mu$ for covariant gauge. Now

$$Z(0) = \int \mathcal{D} A_\mu \; e^{iS(A_\mu)} \int \mathcal{D}\Lambda \; \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right)$$

Change the order of integration & define a new field $A'_\mu = A_\mu^\Lambda$ to write (dropping the prime)

(as in QED $\mathcal{D} A_\mu = \mathcal{D} A_\mu^\Lambda$, $S(A_\mu) = S(A_\mu^\Lambda)$)

$$Z(0) = \underbrace{\int \mathcal{D}\Lambda}_{\text{still } \infty \text{, but an overall factor}} \cdot \int \mathcal{D}A_\mu \; e^{iS'(A_\mu)} \; \delta(G(A_\mu)) \underbrace{\det\left(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\right)}_{\text{we assumed that this is independent of } \Lambda}.$$

still ∞, but an overall factor $\Rightarrow$ cancels in correlators like

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{1}{Z} \cdot \int \mathcal{D}A_\mu \;\; A_\mu(x) A_\nu(y) \; e^{iS'}.$$

a trick: choose $G(A) = \partial_\mu A^\mu - \omega(x) \Rightarrow$

$$\Rightarrow \delta(G(A)) = \delta(\overbrace{\partial_\mu A^\mu}^{\bar{G}(A)} - \omega(x))$$

Nothing else in Z depends on $\omega(x)$ $\Rightarrow$ integrate over $\omega(x)$:

$$1 = \underbrace{N(\xi)}_{\text{norm}} \int \mathcal{D}\omega(x) \; e^{-i\int d^4x \, \frac{\omega^2}{2\xi}}$$

$$\Rightarrow Z(0) = \int \mathcal{D}\Lambda \cdot N(\xi) \int \mathcal{D}A_\mu \; e^{iS'(A_\mu)} \int \mathcal{D}\omega \; e^{-i\int d^4x \, \frac{\omega^2}{2\xi}} \cdot$$

$$\cdot \, \delta\left(\bar{G}(A) - \omega(x)\right) \det\left(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\right) = \int \mathcal{D}\Lambda \cdot N(\xi) \cdot$$

$$\cdot \int \mathcal{D}A_\mu \cdot \det\left(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\right) \cdot e^{iS'(A_\mu)} \cdot e^{-i\int d^4x \, \frac{1}{2\xi}(\bar{G}(A))^2}.$$

$\int \mathcal{D}\Lambda \cdot N(\xi)$ is an unimportant overall factor.

What do we do with $\det\left(\frac{\delta G}{\delta\Lambda}\right)$?

We have

$$Z(0) \sim \int \mathcal{D} A_\mu \, \det\left(\frac{\delta G(A^\wedge)}{\delta \wedge}\right) \cdot e^{i S(A) - i \int d^4x \frac{[G(A)]^2}{2\xi}}$$

We want to put det into the exponent ~ make it a part of the Lagrangian.

Note that $\int\limits_{-\infty}^{\infty} dx \, e^{-a x^2} = \sqrt{\frac{\pi}{a}} \Rightarrow$

$$\Rightarrow \int\limits_{-\infty}^{\infty} dx_1 \ldots dx_n \, e^{-a_1 x_1^2 - \ldots - a_n x_n^2} = (\pi)^{n/2} \cdot \frac{1}{\sqrt{a_1 a_2 \ldots a_n}}$$

$$= \frac{\pi^{n/2}}{\sqrt{\det A}}, \quad A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots a_n \end{pmatrix} \text{ a diagonal matrix.}$$

Similarly, for $\forall A$ get
$$\boxed{\int\limits_{-\infty}^{\infty} d^n x \, e^{-x^T A x} = \frac{\pi^{n/2}}{\sqrt{\det A}}}$$

$\Rightarrow$ can absorb $\frac{1}{\sqrt{\det A}}$ into exponent. But here we have $\det A$!

Grassmann quantities: η is a Grassmann #

$\Rightarrow$ if η is single-component $\Rightarrow f(\eta) = A + B\eta$

$(\eta^2 = 0, \ \eta^3 = 0, \ldots) \Rightarrow \frac{df}{d\eta} = B \Rightarrow \frac{d^2 f}{d\eta^2} = 0$

$\Rightarrow$ no inverse to differentiation?

To define integrals note:

$$\int d\eta\, f(\eta) = \int d\eta\, (A + B\eta) = \Big|^{\eta \to \eta + \Theta} = \int d\eta\, (A + B\eta + B\Theta)$$

$$\Rightarrow \int d\eta \cdot A = \int d\eta\, (A + B\Theta) \Rightarrow \int d\eta = 0$$

$$\int d\eta\, B\eta = B \quad (\text{linear in } B\text{, adjust constant to } 1)$$

$$\Rightarrow \int d\eta \cdot \eta = 1$$

Complex η: $\bar{\eta}$ is c.c. $\int d\eta = \int d\bar{\eta} = 0$, $\int d\eta \cdot \eta = \int d\bar{\eta}\, \bar{\eta} = 1$.

$$\int d\bar{\eta}\, d\eta\, e^{-b\bar{\eta}\eta} = \int d\bar{\eta} \int d\eta\, (1 - b\bar{\eta}\eta) = \int d\bar{\eta} \cdot (+)b\bar{\eta} = 1 b = b$$

Two-component Grassmann #'s: $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, $\bar{\eta} = \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \end{pmatrix}$

$$\Rightarrow \bar{\eta}\eta = \bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2$$

$$(\bar{\eta}\eta)^2 = (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2)^2 = 2\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2$$

$\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}^T M \eta}$ need to find, M is a 2×2 matrix

$$\Rightarrow e^{-\bar{\eta}^T M \eta} = e^{-(\bar{\eta}_1\ \bar{\eta}_2)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}} =$$

$$= \exp\left\{-(\bar{\eta}_1\ \bar{\eta}_2)\begin{pmatrix} M_{11}\eta_1 + M_{12}\eta_2 \\ M_{21}\eta_1 + M_{22}\eta_2 \end{pmatrix}\right\} = \exp\left\{-\left[\bar{\eta}_1(M_{11}\eta_1 + M_{12}\eta_2) + \right.\right.$$

$$+\bar{\eta}_2\left(M_{21}\eta_1+M_{22}\eta_2\right)\Big]\Big\} = 1-\bar{\eta}_1\left(M_{11}\eta_1+M_{12}\eta_2\right)$$

$$-\bar{\eta}_2\left(M_{21}\eta_1+M_{22}\eta_2\right)+\frac{1}{2}\,2\,\bar{\eta}_1\left(M_{11}\eta_1+M_{12}\eta_2\right)\bar{\eta}_2\left(M_{21}\eta_1+\right.$$

$$\left.+M_{22}\eta_2\right) = 1-\bar{\eta}_1\left(M_{11}\eta_1+M_{12}\eta_2\right)-\bar{\eta}_2\left(M_{21}\eta_1+M_{22}\eta_2\right)$$

$$+\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2\left(M_{11}M_{22}-M_{12}M_{21}\right)$$

$$\Rightarrow \int d\bar{\eta}\,d\eta\; e^{-\bar{\eta}^T M\eta} = \int \overbrace{d\bar{\eta}_1\,d\bar{\eta}_2\,d\eta_2\,d\eta_1}^{d\bar{\eta}_1 d\eta_1\, d\bar{\eta}_2 d\eta_2}\;\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2\cdot$$

$$\cdot\left(M_{11}M_{22}-M_{12}M_{21}\right) = \det M.$$

$\Rightarrow$ can show for $\forall$ dimension

$$\int d\bar{\eta}\,d\eta\; e^{-\bar{\eta}^T M\eta} = \det M$$

$$\Rightarrow \det\left[\frac{\delta G(A^\Lambda)}{\delta\Lambda}\right] = \int \mathcal{D}\eta\,\mathcal{D}\bar{\eta}\; e^{-i\int d^4x\,\bar{\eta}^a\frac{\delta G}{\delta\Lambda}\eta^a}.$$

$$\Rightarrow Z(0)\sim\int \mathcal{D}\underbrace{A_\mu}_{\mathcal{D}\eta\mathcal{D}\bar{\eta}}\; e^{iS(A)-i\int d^4x\frac{[\bar{G}(A)]^2}{2\xi}-i\int d^4x\,\bar{\eta}\frac{\delta G}{\delta\Lambda}\eta}$$

η^a ~ Faddeev-Popov ghost, has a color index $a=1,\dots,N^2-1$ for $SU(N)$

<u>Covariant gauge</u>: $\bar{G}(A)=\partial_\mu A^{a\mu}$ $\Rightarrow$ gauge transform is $A_\mu\to\Lambda A_\mu\Lambda^{-1}-\frac{i}{g}(\partial_\mu\Lambda)\Lambda^{-1}$

$\Rightarrow$ for infinitesimal gauge transform:

$$\Lambda = 1 + i\,\alpha^a T^a \quad \Rightarrow \quad A^a_\mu T^a \to (1 + i\,\alpha^a T^a)\, A_\mu \cdot$$

$$\cdot\,(1 - i\,\alpha^b T^b) - \frac{i}{g}\, i T^a (\partial_\mu \alpha^a)(1 - i\,\alpha^b T^b) =$$

$$= A_\mu + i\,[\alpha, A_\mu] + \frac{1}{g}\,\partial_\mu \alpha = T^a A'^a_\mu$$

$$\Rightarrow \quad A'^a_\mu = A^a_\mu + i \cdot i f^{abc} \alpha^b A^c_\mu + \frac{1}{g}\,\partial_\mu \alpha^a$$

$$= A^a_\mu + f^{abc} A^b_\mu \alpha^c + \frac{1}{g}\,\partial_\mu \alpha^a =$$

$$= A^a_\mu + \frac{1}{g}\,\mathcal{D}_\mu \alpha^a, \text{ where } \mathcal{D}_\mu \alpha^a = \partial_\mu \alpha^a + g\, f^{abc} A^b_\mu \alpha^c$$

$$\Rightarrow \quad \frac{\delta G}{\delta \Lambda} = \frac{\delta G}{\delta \alpha} = \frac{\delta\,(\partial_\mu A^{a\mu})}{\delta \alpha} = \partial_\mu \underbrace{\frac{1}{g}}_{\text{absorb by re-defining } \eta} \mathcal{D}^\mu$$

$$\Rightarrow \quad \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2 - \bar{\eta}\, \partial_\mu \mathcal{D}^\mu \eta$$

$$\Rightarrow \quad \boxed{\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{\eta}^a)\, \mathcal{D}^\mu \eta^a}$$

Faddeev - Popov Lagrangian

<u>Light-cone gauge</u>: $\bar{G}(A) = n \cdot A^a = n_\mu A^{a\mu} \quad \Rightarrow$

$$\Rightarrow \quad \frac{\delta G}{\delta \alpha} = n_\mu \frac{\delta A^{a\mu}}{\delta \alpha} = n_\mu \frac{1}{g}\,\mathcal{D}^\mu = \frac{1}{g}\, n_\mu \,(\partial^\mu - ig\underbrace{[A^\mu, \ldots]}_{0})$$

(in the $\xi \to 0$ limit) $\longrightarrow$ as $n \cdot A = 0$

$= \frac{1}{g} n \cdot \partial$ => is A_μ - independent =>

=> ~~again~~ gives only an overall factor in

=> do not need it, no need to introduce the ghost!

=> no ghosts in light-cone gauge!

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(n \cdot A^a)^2$$ LC gauge Lagrangian.

($\xi \to 0$ is implicit)

=> unlike QED case, in QCD in covariant gauge $\underline{\delta G(A^\Lambda)} \over \delta \Lambda$ depends on A_μ and can not be taken out of DA_μ integral => have to introduce the ghost field!