Last time | Faddeev - Popov Quantization

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \Rightarrow \int \mathcal{D}A_\mu \, e^{-\frac{i}{4}\int d^4x (F_{\mu\nu}^a)^2}$$

$\Rightarrow$ Again ∞ gauge degrees of freedom $\Rightarrow$ have to fix gauge, but this time $\det \frac{\delta G(A^\Lambda)}{\delta \Lambda}$ may depend on A_μ.

$$\int \mathcal{D}A_\mu \, e^{i\int d^4x \left(\frac{-1}{4}\right)(F_{\mu\nu}^a)^2} = \underbrace{\left(\int \mathcal{D}\Lambda\right) N(\xi)}_{\text{overall factors}\sim\text{cancel}} \cdot \int \mathcal{D}A_\mu \, \mathcal{D}\bar{\eta} \, \mathcal{D}\eta \cdot e^{i\int d^4x \mathcal{L}_{FP}}$$

$$\boxed{\mathcal{L}_{FP} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{\eta}^a) D^\mu \eta^a}$$

Faddeev - Popov Lagrangian in $\partial_\mu A^{a\mu} = 0$ Lorenz gauge

η^a = ghost field (scalar Grassmann field)

$$\boxed{D_\mu \eta^a = \partial_\mu \eta^a + g f^{abc} A_\mu^b \eta^c}$$ adjoint covariant derivative

Other gauges: light-cone-like gauge: $\boxed{n_\mu A^{a\mu} = 0}$

$$\Rightarrow \boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\xi}(n\cdot A)^2},$$ take $\xi \to 0$ at the end

$\Rightarrow$ no ghosts in such gauges

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^f \left[i \not{D} - m_f \right] q^f - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}$$

Fritzsch, Gell-Mann, Leutwyler ('73)

Gross & Wilczek ('73)

Weinberg ('73)

$f = u, d, s, c, b, t$

Feynman Rules in QCD

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^f[i\gamma^\mu D_\mu - m_f]q^f - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}}$$

However, this Lagrangian is gauge-invariant

$$\begin{cases} A_\mu \to S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1} \\ q \to S q \end{cases}$$

$\Rightarrow$ need to fix the gauge!

(i) Covariant (Lorenz) gauge $\partial_\mu A^{a\mu} = 0$

$\Rightarrow$ to fix the gauge need to introduce the so-called ghost fields:

$$\mathcal{L}_{QCD}^{cov.gauge} = \bar{q}^f[i\gamma^\mu D_\mu - m_f]q^f - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})(\partial_\nu A^{a\nu}) + \partial_\mu \bar{\eta}^a \mathcal{D}^\mu \eta^a$$

η^a is a scalar field ~ Faddeev-Popov ghost

η^a is an anti-commuting field (Grassmann variables) (quantized like a fermion) $\Rightarrow$ unphysical $\Rightarrow$ ghosts

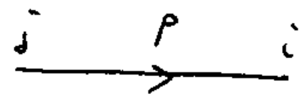
$\bar{\eta}^a$ is c.c. of η ; $D_\mu = \partial_\mu - ig A_\mu$

$$\eta = \sum_{a=1}^{8} T^a \eta^a , \quad D_\mu \eta = \partial_\mu \eta - ig \underbrace{[A_\mu, \eta]}_{\text{note the commutator!}}$$

$$D_\mu \eta^a = \partial_\mu \eta^a + g f^{abc} A_\mu^b \eta^c$$

Feynman Rules:

Quark Propagator:



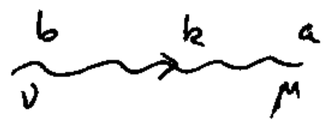
$$\frac{i}{\gamma \cdot p - m} \delta_{ij}$$

$$= \frac{i(\gamma \cdot p + m)}{p^2 - m^2 + i\varepsilon} \delta_{ij}$$

Ghost Propagator:

$$\frac{i}{k^2 + i\varepsilon} \delta_{ab}$$

Gluon Propagator:

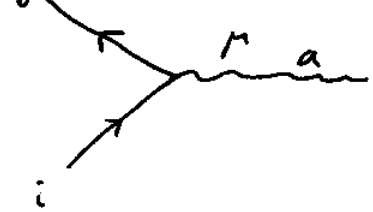


$$\frac{-i}{k^2 + i\varepsilon} \delta^{ab} \left[g_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right]$$

$\xi = 0$ Landau gauge

$\xi = 1$ Feynman gauge.

Quark-Gluon Vertex:



$$ig \gamma^\mu (T^a)_{ji}$$

Other interaction vertices are less trivial:

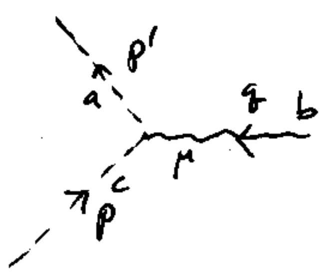
$$\partial_\mu \bar\eta\, D^\mu \eta = \partial_\mu \bar\eta\, \partial^\mu \eta \underbrace{- ig\, \partial_\mu \bar\eta\, [A^\mu, \eta]}_{\text{ghost-gluon interaction}}$$

$$\Rightarrow \mathcal{L}_{gG} = -ig\, \partial_\mu \bar\eta\, [A^\mu, \eta] = \cancel{-i} g\, \partial_\mu \bar\eta^a\, \cancel{i} f^{abc} A^{b\mu} \eta^c$$

$$= g \left(\partial_\mu \bar\eta^a\right) f^{abc} A^{b\mu} \eta^c$$

When contracting with $\eta(y)$: $\overbrace{\eta(y)\;\; g f^{abc} \big(\underbrace{\partial_\mu}_{ip'_\mu} \bar\eta^a_{(x)}\big)}^{= e^{-ip'\cdot(y-x)}} A^{b\mu} \eta^c$

$$\Rightarrow i g f^{abc} p'_\mu \otimes \overset{\text{from } iS}{i} \Rightarrow -g f^{abc} p'_\mu = g f^{bac} p'_\mu$$



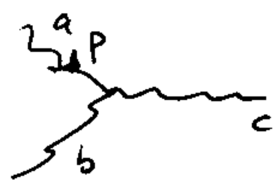
is the contribution of ghost - gluon vertex

3 - gluon vertex: $-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) \cdot g \cdot$

$$f^{abc} A^b_\mu A^c_\nu = -g\, \partial_\mu A^a_\nu\, f^{abc} A^b_\mu A^c_\nu$$

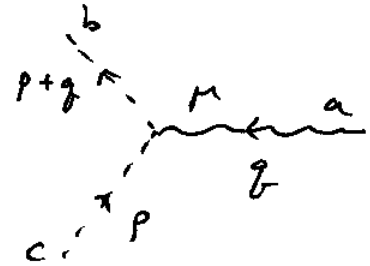
$\Rightarrow$ contracting with $A^d_\rho(y)$: $-g f^{abc} A^b_\mu A^c_\nu \underbrace{\partial_\mu}_{-ip_\mu} \overbrace{A^a_\nu(x) A^d_\rho(y)}^{\sim e^{-ip\cdot(x-y)}}$

$$\Rightarrow \text{get terms like } \overset{\text{from } iS}{i} g f^{abc} p_\mu = g f^{bac} p_\mu + \dots$$



$\Rightarrow$ let us summarize all this:

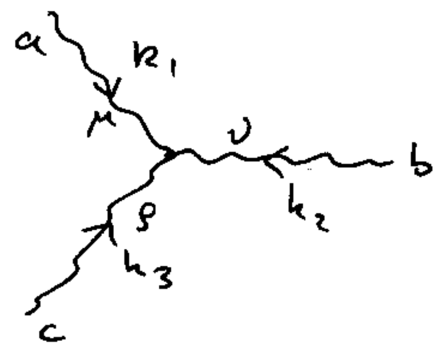
Ghost-gluon Vertex:



$$g\,(p+q)_\mu\, f^{abc}$$

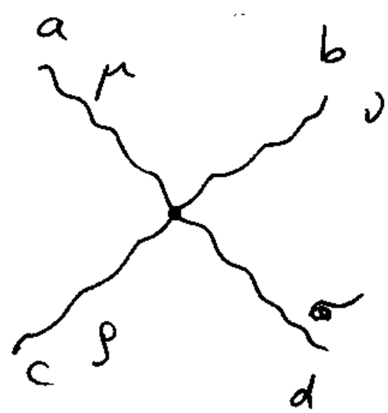
(counter-clockwise)

3-Gluon Vertex:



$$-g\, f^{abc}\Big[(k_1-k_3)_\nu\, g_{\mu\rho}$$
$$+\,(k_2-k_1)_\rho\, g_{\mu\nu} + (k_3-k_2)_\mu\, g_{\nu\rho}\Big]$$

4-Gluon Vertex:



$$-ig^2\Big[f^{abe} f^{cde} \cdot (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$
$$+ f^{ace} f^{bde}(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho})$$
$$+ f^{ade} f^{bce}(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})\Big].$$

~ same as QED for external fermions, bosons (no external ghosts), internal integrals, "−" for each fermion (or ghost) loop.

(ii) <u>Light-cone gauge</u>

Define light-cone variables: $A^{\pm} = \dfrac{A^0 \pm A^3}{\sqrt{2}}$

(choose a "preferred direction" $\sim x^3$)

$A^+ = 0$ gauge is called the light-cone (LC) gauge

Write the gauge condition as

$$\eta \cdot A = 0 \quad \text{with} \quad \eta^- = 1,\ \eta^+ = 0,\ \eta^1 = \eta^2 = 0$$

$$A_\mu B^\mu = A^+ B^- + A^- B^+ - A^1 B^1 - A^2 B^2$$
(check)

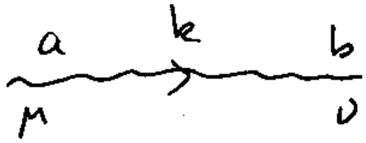
$$\eta \cdot A = \underset{0}{\underset{\parallel}{\eta^+}} A^- + \underset{1}{\underset{\parallel}{\eta^-}} A^+ - \underset{0}{\underset{\parallel}{\eta^1}} A^1 - \underset{0}{\underset{\parallel}{\eta^2}} A^2 = A^+$$

$\Rightarrow$ there is <u>no ghost</u> in LC gauge!

<u>Feynman rules</u>: the same, but no ghost

$\Rightarrow$ no ghost propagator, no ghost-gluon vertex

$\Rightarrow$ gluon propagator is different:



$$\frac{-i}{k^2 + i\varepsilon}\, \delta^{ab} \left[g_{\mu\nu} - \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta \cdot k} \right].$$

Renormalization of QCD.

We'll work in $\partial_\mu A^\mu = 0$ covariant gauge.

Bare QCD Lagrangian is:

$$\boxed{\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{0\mu\nu} F_0^{a\mu\nu} - \frac{1}{2\xi_0}(\partial_\mu A_0^{a\mu})^2 + (\partial_\mu \bar{\eta}_0^a)\mathcal{D}^\mu \eta_0^a + \bar{\psi}_0[i\not{D} - m_0]\psi_0.}$$

where

$$F^a_{0\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} + g_0 f^{abc} A^b_{0\mu} A^c_{0\nu}$$

$$D_\mu = \partial_\mu - i g_0 A_{0\mu}\,, \quad \mathcal{D}_\mu = \partial_\mu - i g_0 [A_{0\mu}, \ldots]$$

As for QED define physical fields:

$$\boxed{\psi = \frac{1}{\sqrt{Z_2}}\psi_0\,, \quad A_\mu = \frac{1}{\sqrt{Z_3}} A_{0\mu}}$$

and the same for the ghost: $\boxed{\eta^a = \frac{1}{\sqrt{Z_2^\eta}}\eta_0^a}$

$(Z_2^\eta \neq Z_2)$

Note that $\mathcal{D} A_\mu = \mathcal{D} A_{0\mu}$ $\Rightarrow$ rescaling does not change the measure of Feynman integral.

We write:

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{\eta}^a) D^\mu \eta^a +$$

$$+ \bar{\psi}[i \not{D} - m]\psi + \bar{\psi}[i \not{\partial} \delta_2 - \delta_m]\psi +$$

$$+ g \delta_1 \bar{\psi} \not{A} \psi + \delta_2^\eta \partial_\mu \bar{\eta}^a \partial^\mu \eta^a + g\, \delta_1^\eta \partial_\mu \bar{\eta}^a f^{abc} A^b_\mu \eta^c$$

$$- \frac{1}{4} \delta_3 (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - g\, \delta_1^{3g} f^{abc} (\partial_\mu A^a_\nu) A^b_\mu A^c_\nu -$$

$$- \frac{g^2}{4} \delta_1^{4g} f^{eab} A^a_\mu A^b_\nu f^{ecd} A^c_\mu A^d_\nu$$

where $\delta_2 = Z_2 - 1$, $\delta_3 = Z_3 - 1$, $\delta_2^\eta = Z_2^\eta - 1$,

$$\delta_m = Z_2 m_0 - m, \quad \delta_1 = \frac{g_0}{g} Z_2 Z_3^{1/2} - 1 = Z_1 - 1$$

$$\delta_1^\eta = \underbrace{\frac{g_0}{g} Z_2^\eta Z_3^{1/2}}_{= Z_1^\eta} - 1, \quad \delta_1^{3g} = \frac{g_0}{g} Z_3^{3/2} - 1,$$

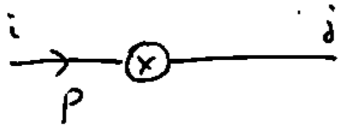
$$\delta_1^{4g} = \frac{g_0^2}{g^2} (Z_3)^2 - 1, \quad \xi = \frac{\xi_0}{Z_3}$$

8 counterterms! However, there are relations between the counterterm coefficients, as there are 5 indep. parameters: $Z_2, Z_3, Z_2^\eta, \overset{\text{or } Z_1,\, \delta_m}{\widetilde{g_0, m_0}}$.

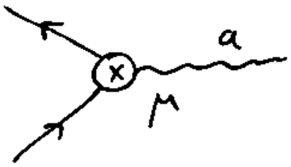
(Slavnov-Taylor identities: like Ward identities in QED which had $Z_1 = Z_2$, but more subtle)

Additional Feynman rules due to counterterms:

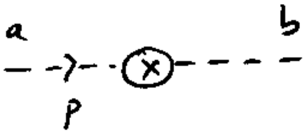
$$= -i\,\delta^{ab}\,\delta_3\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)$$



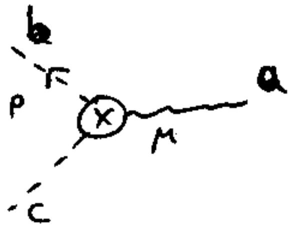
$$= i\left(\delta_2 \not p - \delta_m\right)\delta_{ij}$$ ~ color indeces



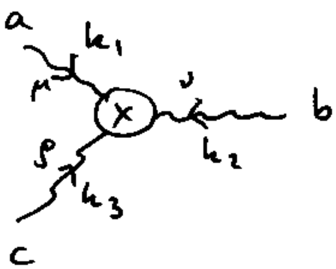
$$= i g\, T^a \gamma^\mu \delta_1$$



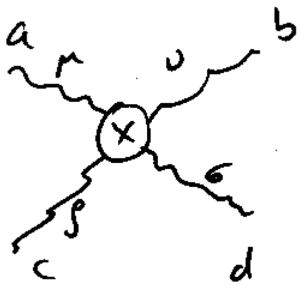
$$= i\, p^2\, \delta_2^\eta\, \delta^{ab}$$



$$= g\, p_\mu\, f^{abc}\, \delta_1^\eta$$



$$= -\delta_1^{3g} \cdot g\, f^{abc}\Big[(k_1-k_3)_\nu g_{\mu\rho} + (k_2-k_1)_\rho g_{\mu\nu} + (k_3-k_2)_\mu g_{\nu\rho}\Big]$$



$$= -i\,\delta_1^{4g} g^2\Big[f^{abe} f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ace} f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ade} f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\Big]$$