Last time | Running of QCD Coupling and Asymptotic Freedom (cont'd)

$\Rightarrow$ QCD beta-function:

$$\beta(\alpha) = \lim_{\varepsilon \to 0} \left\{ \alpha\, \varepsilon \left[\delta_1 - \delta_2 - \frac{1}{2}\delta_3 \right] \right\}$$

δ_2 comes from quark self-energy: 

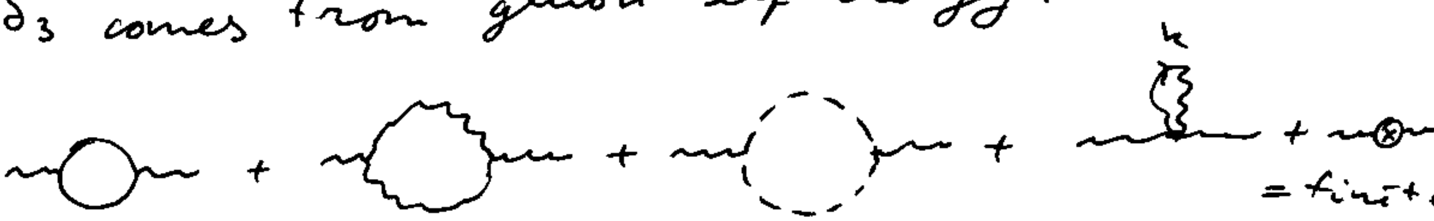
$\Rightarrow$ using QED result got $\delta_2 = -\frac{\alpha C_F}{2\pi\varepsilon}$, $C_F = \frac{N_c^2 - 1}{2N_c}$.

δ_1 comes from vertex corrections:

$= \text{finite} \Rightarrow \delta_1 = -\frac{\alpha}{4\pi\varepsilon}\left[3N_c - \frac{1}{N_c}\right]$.

QED analogue   picked UV div.

δ_3 comes from gluon self-energy:



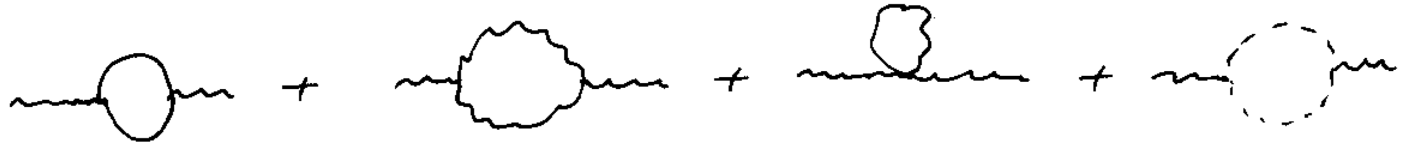
$= \text{finite}$

$\delta_3^f = -\frac{\alpha N_f}{3\pi\varepsilon}$ (QED analogue, N_f = # flavors)

to be calculated now

$\int \frac{d^d k}{k^2} = 0 \Rightarrow$ does not contribute to δ_3.

Finally, we also need δ_3. To find it we calculate gluon self-energy up to $o(\alpha)$:



$+$  $=$ finite

Start with quark loop:

$$= \begin{pmatrix}\text{same as in}\\ \text{QED}\end{pmatrix} \otimes \begin{pmatrix}\text{color}\\ \text{factor}\end{pmatrix} \otimes N_f$$

$\begin{pmatrix}\text{same as in QED}\end{pmatrix} = -\dfrac{\alpha}{3\pi}\dfrac{2}{\varepsilon}$

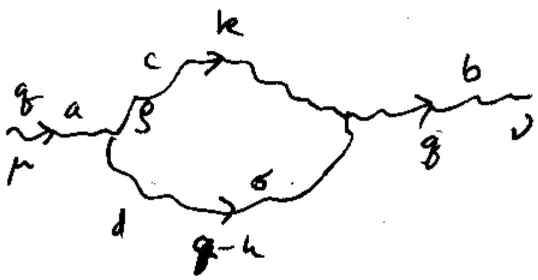
$\begin{pmatrix}\text{color factor}\end{pmatrix} = tr\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$

N_f = # of quark flavors

$$\Rightarrow \boxed{\delta_3^{f} = -\frac{\alpha}{3\pi}\frac{1}{\varepsilon}N_f}$$

(f = quarks)

Gluon loop:



$$= g^2 \cdot \underbrace{f^{acd} f^{cbd}}_{-N_c\delta^{ab}} \cdot \int \frac{d^dk}{(2\pi)^d}\,\frac{-i}{k^2}\,\frac{-i}{(q-k)^2}$$

$$\cdot\left[(2q-k)_\rho g_{\mu\sigma} + (-k-q)_\sigma g_{\mu\rho} + (2k-q)_\mu g_{\rho\sigma}\right]\cdot$$

$$\cdot\left[(2k-q)_\nu g_{\rho\sigma} + (-q-k)_\sigma g_{\rho\nu} + (2q-k)_\rho g_{\nu\sigma}\right]\cdot\frac{1}{2}$$

($\frac{1}{2}$ = symmetry factor)

$$= \frac{g^2 N_c}{2}\delta^{ab}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}\cdot\frac{1}{(q-k)^2}\,[\ldots][\ldots]$$

$$[\dots][\dots] = (2q-k)_\mu (2k-q)_\nu - (2q-k)_\nu (q+k)_\mu +$$

$$-(2q-k)^2 g_{\mu\nu} - (k+q)_\mu (2k-q)_\nu + (k+q)^2 g_{\mu\nu} - (2q-k)_\mu (k+q)_\nu$$

$$+ \underset{g^\mu{}_\mu}{\overset{}{d}}\,(2k-q)_\mu (2k-q)_\nu - (2k-q)_\mu (k+q)_\nu + (2k-q)_\mu (2q-k)_\nu =$$

$$= g_{\mu\nu}\left[5q^2 - 2q\cdot k + 2k^2\right] + \cancel{(2k-q)_\nu (q-2k)_\mu} + (2q-k)_\nu \cdot$$

$$\cdot (k-2q)_\mu + \overset{d-1}{\cancel{d}}\,(2k-q)_\mu (2k-q)_\nu - (k+q)_\nu (q+k)_\mu =$$

$$= g_{\mu\nu}\left[5q^2 - 2q\cdot k + 2k^2\right] + 2q_\nu k_\mu - 4q_\mu q_\nu - k_\mu k_\nu + 2q_\mu k_\nu$$

$$+4(d-1)k_\mu k_\nu - 2(d-1)k_\mu q_\nu - 2(d-1)q_\mu k_\nu + (d-1)q_\mu q_\nu - k_\mu k_\nu - k_\mu q_\nu - k_\nu q_\mu - q_\mu q_\nu$$

$$= g_{\mu\nu}\left[5q^2 - 2q\cdot k + 2k^2\right] + (d-6)q_\mu q_\nu + 2(2d-3)k_\mu k_\nu + (3-2d)q_\mu k_\nu + (3-2d)k_\mu q_\nu$$

$\Rightarrow$ 

$$= \frac{g^2 N_c}{2}\delta^{ab}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[\underbrace{(k-xq)}_{\text{new } k}{}^2 + x(1-x)q^2\right]^2}$$

$$\cdot\left\{ g_{\mu\nu}\left[5q^2 - 2q\cdot k + 2k^2\right] - 2q_\mu q_\nu + 2(2d-3)k_\mu k_\nu - 5q_\mu k_\nu - 5k_\mu q_\nu \right\}$$

$$= \frac{g^2 N_c}{2}\delta^{ab}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 + x(1-x)q^2\right]^2} \left\{ g_{\mu\nu}\left[5q^2 - 2xq^2 + 2k^2\right.\right.$$

$$\left.\left. + 2x^2q^2\right] - 2q_\mu q_\nu + 2(2d-3)k_\mu k_\nu + 10x^2 q_\mu q_\nu - 10x\, q_\mu q_\nu \right\} =$$

$$= \frac{g^2 N_c}{2}\delta^{ab}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 + x(1-x)q^2\right]^2} \left\{ g_{\mu\nu}\left[q^2(5-2x+2x^2) + 2k^2\right]\right.$$

$$\left. - q_\mu q_\nu\, 2\left[1 - 5x^2 + 5x\right] + \frac{2(2d-3)}{d}k^2 g_{\mu\nu} \right\} =$$

$$= \frac{g^2 N_c}{2} i \delta^{ab} \int_0^1 dx \cdot \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{\left[k_E^2 - x(1-x)q^2\right]^2} \Big\{ -k_E^2\, 6\left(1-\frac{1}{d}\right) g_{\mu\nu}$$

$$+ g_{\mu\nu}\, q^2 \left(5 - 2x + 2x^2\right) - 2 q_\mu q_\nu \left(1 + 5x - 5x^2\right) \Big\} =$$

$$= \Bigg(\text{using} \int \frac{d^d k_E}{(2\pi)^d} \frac{k_E^2}{\left[k_E^2 + \Lambda^2\right]^2} = \frac{1}{(4\pi)^{d/2}} \cdot \frac{d}{2} \cdot \Gamma\left(1 - \frac{d}{2}\right) \left(\Lambda^2\right)^{\frac{d}{2}-1}$$

$$\text{and} \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{\left[k_E^2 + \Lambda^2\right]^2} = \frac{1}{(4\pi)^{d/2}} \left(\Lambda^2\right)^{\frac{d}{2}-2} \Gamma\left(2 - \frac{d}{2}\right) \Bigg)$$

$$= i \frac{g^2 N_c}{2} \delta^{ab} \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \left[-x(1-x)q^2\right]^{\frac{d}{2}-2} \cdot \Big\{ -6\left(1-\frac{1}{d}\right)\frac{d}{2} \cdot g_{\mu\nu}$$

$$\cdot \Gamma\left(1-\frac{d}{2}\right)(-)x(1-x)q^2 + \Gamma\left(2-\frac{d}{2}\right)\Big[g_{\mu\nu} q^2 \left(5-2x+2x^2\right) - 2 q_\mu q_\nu \cdot$$

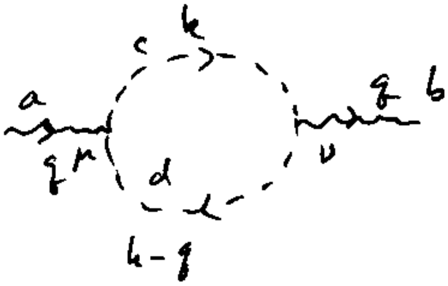
$$\cdot \left(1+5x-5x^2\right)\Big]\Big\} = \Big|^{\varepsilon = 4-d} = i\delta^{ab} \frac{\alpha N_c}{8\pi} \cdot \int_0^1 dx \Big\{ +x(1-x) q^2 \cdot 9 \cdot$$

$$\cdot g_{\mu\nu} \underbrace{\Gamma\left(-1 + \frac{\varepsilon}{2}\right)}_{-\frac{2}{\varepsilon}} + \frac{2}{\varepsilon}\Big[g_{\mu\nu} q^2 \left(5 - 2x + 2x^2\right) - 2 q_\mu q_\nu \left(1 + 5x - 5x^2\right)\Big]\Big\} =$$

$$= i\delta^{ab} \frac{\alpha N_c}{4\pi} \cdot \frac{1}{\varepsilon} \left\{ -\frac{3}{2} q^2 g_{\mu\nu} + \frac{14}{3} q^2 g_{\mu\nu} - \frac{11}{3} q_\mu q_\nu \right\}$$

$$= i\delta^{ab} \frac{\alpha N_c}{4\pi} \frac{1}{\varepsilon} \left\{ \frac{19}{6} q^2 g_{\mu\nu} - \frac{11}{3} q_\mu q_\nu \right\} + \text{finite.}$$

Ghost loop:



$$\overbrace{=-g^2}^{\text{ghost loop}}\int\frac{d^dk}{(2\pi)^d}\cdot k_\mu (k-q)_\nu\cdot\frac{i}{k^2}\,\frac{i}{(k-q)^2}$$

$$\underbrace{f^{adc}\cdot f^{bcd}}_{-N_c\delta^{ab}}$$

$$=-g^2N_c\delta^{ab}\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\;\frac{\overbrace{k_\mu(k-q)_\nu}^{k_\mu k_\nu-k_\mu q_\nu}}{[\underbrace{(1-x)k^2+x(k-q)^2}_{(k-xq)^2+x(1-x)q^2}]^2}=$$

$$=-g^2N_c\delta^{ab}\int_0^1dx\int\frac{d^dk}{(2\pi)^d}\;\frac{k_\mu k_\nu+x^2q_\mu q_\nu-xq_\mu q_\nu}{[k^2+x(1-x)q^2]^2}=$$

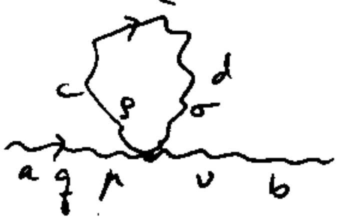
$$=-ig^2N_c\delta^{ab}\int_0^1dx\int\frac{d^dk_E}{(2\pi)^d}\;\frac{-\frac{k_E^2}{d}g_{\mu\nu}-x(1-x)q_\mu q_\nu}{[k_E^2-x(1-x)q^2]^2}=$$

$$=-ig^2N_c\delta^{ab}\int_0^1dx\,\frac{1}{(4\pi)^{d/2}}\cdot\left[x(1-x)q^2\right]^{\frac d2-2}\cdot\Big\{-\frac{1}{\cancel{d}}g_{\mu\nu}\overset{-x(1-x)q^2}{\vphantom{x}}\frac{\cancel{d}}{2}\Gamma\!\left(1-\frac d2\right)$$

$$-x(1-x)q_\mu q_\nu\,\Gamma\!\left(2-\frac d2\right)\Big\}=\Big|_{d=4-\varepsilon}=-i\frac{\alpha N_c}{4\pi}\delta^{ab}\int_0^1dx\cdot$$

$$\cdot\Big\{\frac1\varepsilon g_{\mu\nu}\overset{-x(1-x)q^2}{\vphantom{x}}-x(1-x)q_\mu q_\nu\frac{2}{\varepsilon}\Big\}=$$

$$\boxed{= -i\,\frac{\alpha N_c}{4\pi}\,\delta^{ab}\,\frac{1}{\varepsilon}\left[-q^2 g_{\mu\nu} - 2 q_\mu q_\nu\right]\cdot\frac{1}{6}}$$



$$= \underbrace{\frac{1}{2}}_{\text{Symmetry factor}} \int \frac{d^dk}{(2\pi)^d}\,\frac{-i g_{\rho\sigma}}{k^2+i\varepsilon}\,\delta_{cd}\,(-ig^2)\Big[f^{abe}\underbrace{\cancel{f^{cde}}}_{0}$$

$$\cdot\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)$$

$$+ f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\Big] = \frac{-g^2}{2}N_c\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}$$

$$\delta^{ab}\cdot g_{\rho\sigma}\left[\,2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}\right] = -\frac{g^2 N_c}{2}\,\delta^{ab}\cdot$$

$$\cdot\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}\cdot\left[2d\,g^{\mu\nu} - 2g^{\mu\nu}\right] = -g^2 N_c\,\delta^{ab}(d-1)g^{\mu\nu}\cdot$$

$$\cdot\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}\frac{(q-k)^2}{(q-k)^2} = -g^2N_c\,\delta^{ab}(d-1)g^{\mu\nu}\cdot\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}$$

$$\cdot\frac{1}{\left\{(k-xq)^2 + x(1-x)q^2\right\}^2}\cdot\left(q^2 - 2q\cdot k + k^2\right) = -g^2N_c\,\delta^{ab}(d-1)g^{\mu\nu}\cdot$$

$$\cdot\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left[k^2 + x(1-x)q^2\right]^2}\left(q^2 - 2xq^2 + k^2 + x^2q^2\right) = -ig^2N_c\,\delta^{ab}\cdot$$

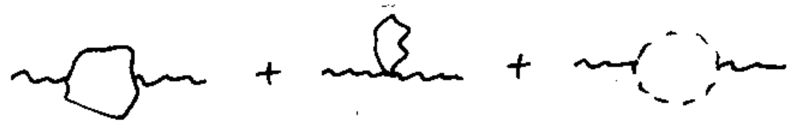
$$\cdot(d-1)g^{\mu\nu}\int_0^1 dx\int\frac{d^dk_E}{(2\pi)^d}\;\frac{-k_E^2 + q^2(1-2x+x^2)}{\left[-k_E^2 + x(1-x)q^2\right]^2} =$$

$$= -i g^2 N_c \delta^{ab} (d-1) g^{\mu\nu} \int_0^1 dx \cdot \frac{1}{(4\pi)^{d/2}} \left[-x(1-x)q^2\right]^{\frac{d}{2}-2} \cdot$$

$$\cdot \left\{ + \frac{d}{2} \Gamma\left(1-\frac{d}{2}\right) (-1) x(1-x) q^2 + q^2 (1-x)^2 \Gamma\left(2-\frac{d}{2}\right) \right\} = \Big|_{d=4-\varepsilon}$$

$$= -i \frac{\alpha N_c}{4\pi} \cdot 3 \, \delta^{ab} g^{\mu\nu} \left\{ -\frac{4}{\varepsilon} \cdot \frac{1}{6} q^2 + \frac{2}{\varepsilon} q^2 \frac{1}{3} \right\} = 0$$

$$\Rightarrow \int \frac{d^d k}{k^2} = 0 .$$

$\Rightarrow$  $= i \, \delta^{ab} \frac{1}{\varepsilon} \frac{\alpha N_c}{4\pi} \cdot \frac{10}{3} \cdot$

$$\cdot \left[q^2 g_{\mu\nu} - q_\mu q_\nu \right]$$

$\Rightarrow$ contrib. to δ_3 is $\delta_3^g = \frac{1}{\varepsilon} \frac{\alpha N_c}{4\pi} \cdot \frac{10}{3}$

$$\Rightarrow \delta_3 = \delta_3^g + \delta_3^f = \frac{\alpha}{4\pi} \frac{1}{\varepsilon} \left[\frac{10}{3} N_c - \frac{4}{3} N_f \right]$$

$$\beta_{QCD}(\alpha) = \alpha \lim_{\varepsilon \to 0} \left\{ \varepsilon \left[\delta_1 - \delta_2 - \frac{1}{2} \delta_3 \right] \right\} = \alpha \cdot \frac{\alpha}{4\pi} \cdot$$

$$\cdot \left\{ -3 N_c + \frac{1}{N_c} + \underbrace{2 C_F}_{N_c - \frac{1}{N_c}} - \frac{5}{3} N_c + \frac{2}{3} N_f \right\} = \frac{\alpha^2}{4\pi} \cdot \left\{ -\frac{11}{3} N_c + \frac{2}{3} N_f \right\}$$

$$\Rightarrow \beta_{QCD}(\alpha) = -\frac{\alpha^2}{12\pi} \left[11 N_c - 2 N_f \right] .$$

Write $\beta_{QCD}(\alpha) = -\beta_2 \alpha^2$ with $\beta_2 = \frac{11 N_c - 2 N_f}{12\pi}$.

In QCD $N_c = 3$, $N_f = 6$ $\Rightarrow$ $\beta_2 = \frac{33-12}{12\pi} = \frac{7}{4\pi} > 0$

$\Rightarrow$ $\boxed{\beta_{QCD}(\alpha) < 0}$ $\Rightarrow$ negative β-function!

(Very unusual, but typical for non-abelian theories.)

$$\boxed{\alpha_s(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln \frac{Q^2}{\mu^2}}} = \frac{1}{\beta_2 \ln \frac{Q^2}{\Lambda^2} + \underbrace{\frac{1}{\alpha_\mu} - \beta_2 \ln \frac{\mu^2}{\Lambda^2}}_{=0}}$$

define Λ^2 by requiring that this is zero

$$\Rightarrow \boxed{\Lambda^2_{QCD} = \mu^2 e^{-\frac{1}{\beta_2 \alpha_\mu}}}$$

fundamental scale of QCD, μ^2-independent

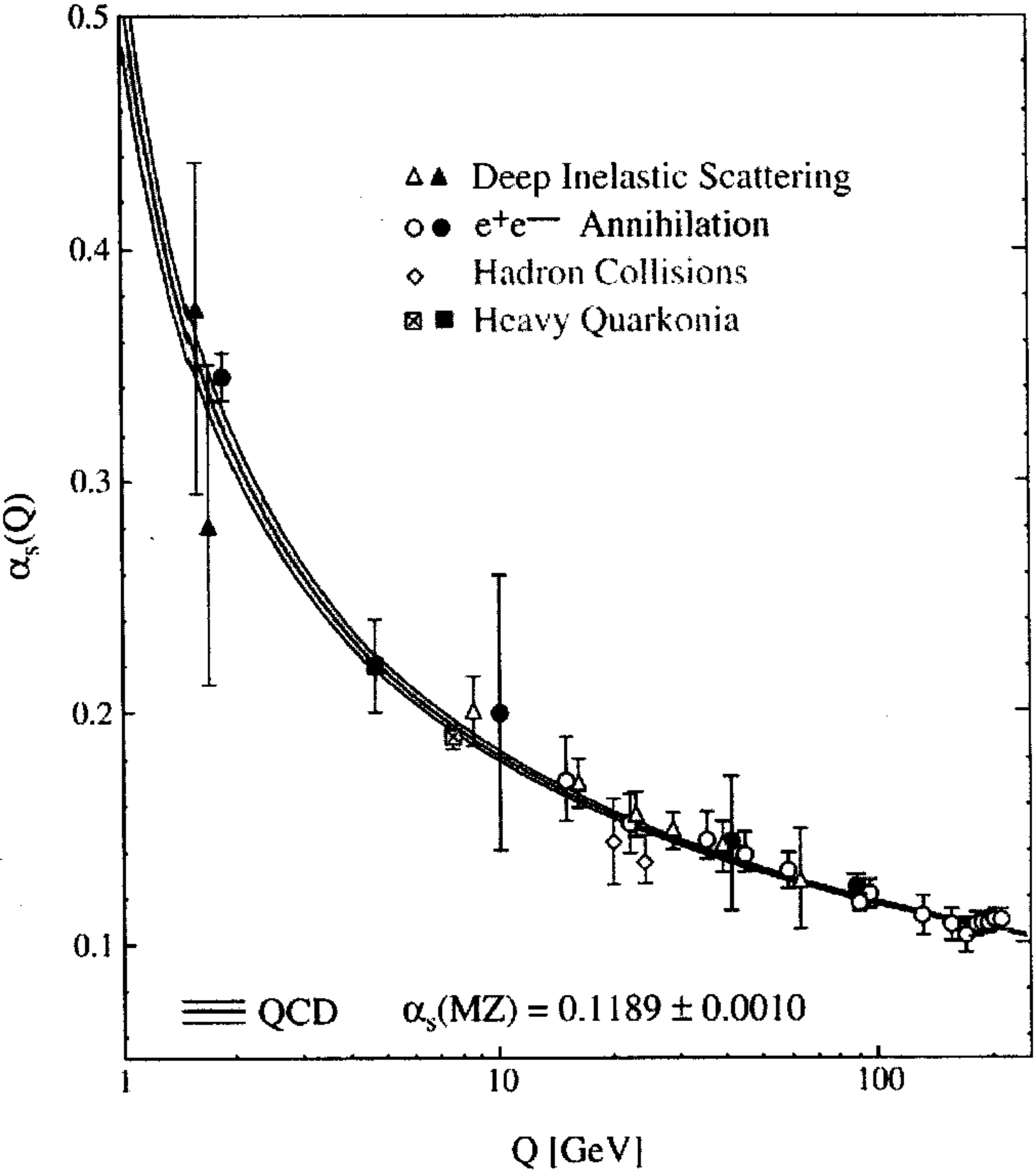
$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu}\right] \Lambda^2_{QCD} = 0$.

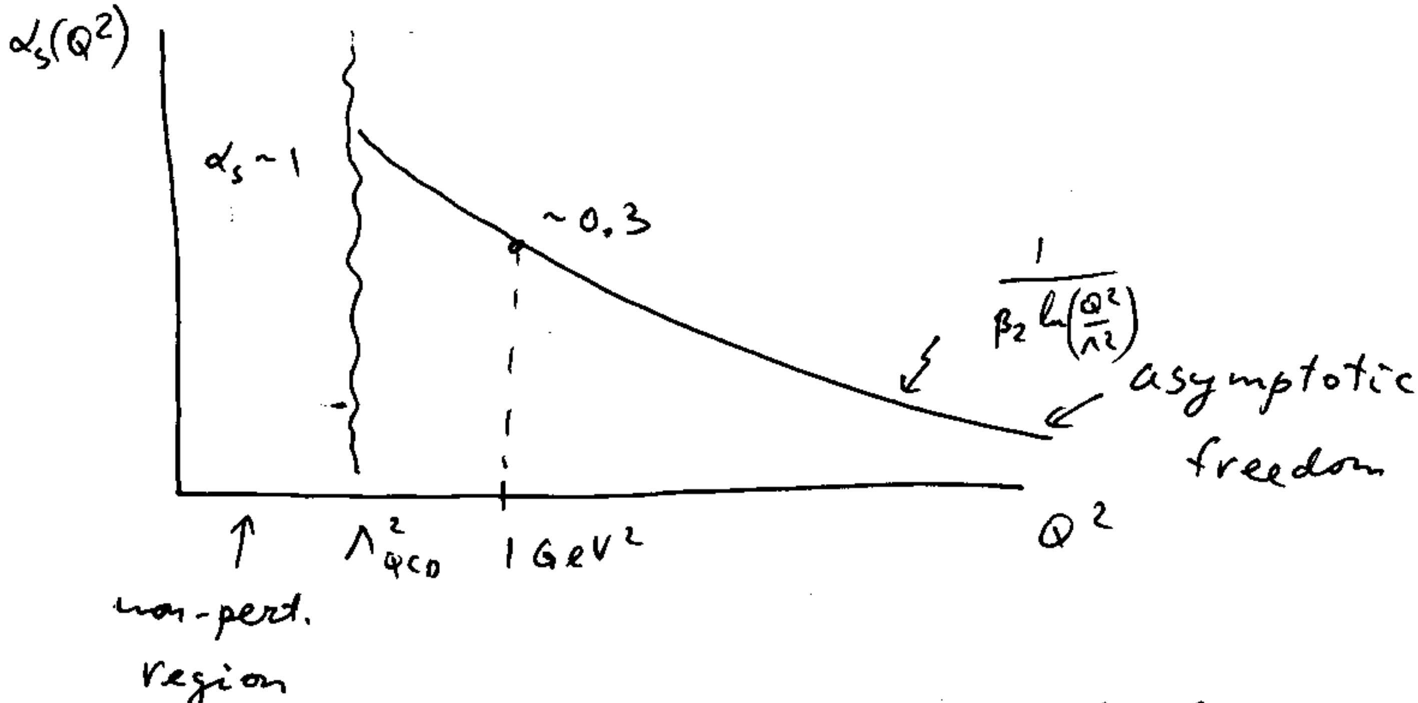
$\Lambda_{QCD} \simeq 200$ MeV (depends on renormalization scheme)

$$\boxed{\alpha_s(Q^2) = \frac{1}{\beta_2 \ln\left(\frac{Q^2}{\Lambda^2_{QCD}}\right)}}$$

$\Rightarrow$ at large-Q^2/short distances $\alpha_s(Q^2) \ll 1$

$\Rightarrow$ Asymptotic freedom (Gross, Politzer, Wilczek '73)


Deep Inelastic Scattering
e+e− Annihilation
Hadron Collisions
Heavy Quarkonia
QCD
αs(MZ) = 0.1189 ± 0.0010
αs(Q)
Q [GeV]
0.5
0.4
0.3
0.2
0.1
1
10
100



$\Rightarrow$ at short distances quarks and gluons are asymptotically free, they hardly interact with each other!

$\Rightarrow$ $\alpha_s(Q^2)$ has Landau pole as well, but unlike QED it is in the IR, at $Q^2 = \Lambda^2_{QCD} \simeq (200\,MeV)^2$

Likely there is no new physics there $\Rightarrow$

$\Rightarrow$ QCD itself remedies the problem through <u>non-perturbative</u> corrections!