Last time: We finished calculating QCD beta-function at one-loop order:

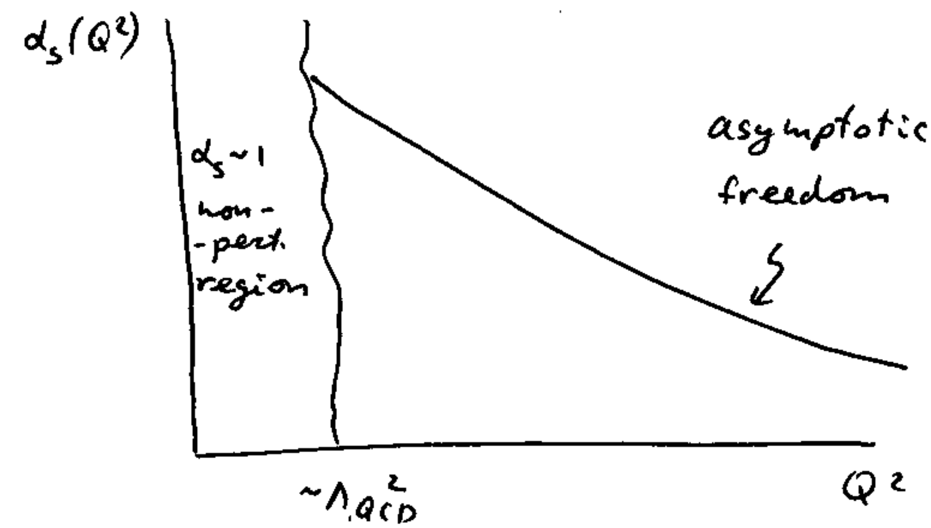
$$\boxed{\beta_{QCD}(\alpha) = -\beta_2 \alpha^2}, \quad \beta_2 = \frac{11 N_c - 2 N_f}{12\pi}$$

Running of QCD coupling:

$$\boxed{\alpha_s(Q^2) = \frac{\alpha_\mu}{1 + \beta_2 \alpha_\mu \ln Q^2/\mu^2} = \frac{1}{\beta_2 \ln \frac{Q^2}{\Lambda_{QCD}^2}}}$$

$\Lambda_{QCD} \gtrsim 200$ MeV fundamental scale of QCD

Landau pole (hopefully fixed by non-perturbative physics)



Asymptotic freedom: quarks and gluons are weakly-interacting ("free") at short distances/large-Q^2.

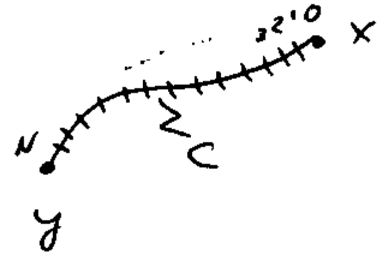
(Gross, Politzer & Wilczek, 1973)

Wilson lines, loops & Heavy Quark Potential

Def. Wilson line:

$$\boxed{W_c(x,y) \equiv P_c \exp\left\{ ig \int_y^x dx'_\mu A^\mu(x') \right\}}$$

Def where a path-ordered exponent is defined as follows. Cut the path connecting y & x along C into slices (W_C depends on C!).



Then

$$\boxed{P_c \exp\left\{ ig \int_y^x dx'_\mu A^\mu(x') \right\} = \prod_{i=1}^{N} \left[1 + ig\, \Delta x_i^\mu A_\mu(x_i) \right]}.$$

$$(x_0^\mu = x^\mu,\ x_N^\mu = y^\mu), \quad \Delta x_i^\mu = x_{i-1}^\mu - x_i^\mu$$

Under gauge transform $A_\mu(x_i) \to S(x_i) A_\mu(x_i) S^{-1}(x_i) - \frac{i}{g}\left(\partial_\mu S(x_i)\right) S^{-1}(x_i)$

$$\Rightarrow W_c(x,y) \to \prod_{i=1}^{N} \left[1 + ig\, \Delta x_i^\mu \left(S(x_i) A_\mu(x_i) S^{-1}(x_i) - \frac{i}{g}\left(\partial_\mu S(x_i)\right) S^{-1}(x_i) \right) \right] =$$

| use $S(x_{i-1}) = S(x_i) + \Delta x_i^\mu \partial_\mu S(x_i)$ and neglect $o(\Delta x^2)$ terms in each factor.

$$= \prod_{i=1}^{N} \Big[1 + ig\Big(\Delta x_i^\mu \, \mathcal{S}(x_{i-1}) \, A_\mu(x_i) \, \mathcal{S}^{-1}(x_i) - \frac{i}{g} \Big(\mathcal{S}(x_{i-1}) -$$

$$- \mathcal{S}(x_i) \Big) \, \mathcal{S}^{-1}(x_i) \Big] = \prod_{i=1}^{N} \Big[\cancel{1} + ig\Big(\Delta x_i^\mu \, \mathcal{S}(x_{i-1}) A_\mu(x_i) \mathcal{S}^{-1}(x_i)$$

$$- \frac{i}{g} \, \mathcal{S}(x_{i-1}) \, \mathcal{S}^{-1}(x_i) + \cancel{\frac{i}{g}} \Big) \Big] = \prod_{i=1}^{N} \mathcal{S}(x_{i-1}) \cdot$$

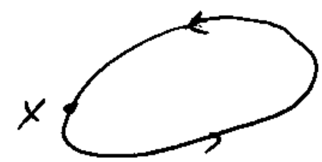
$$\cdot \Big[1 + ig \, \Delta x_i^\mu A_\mu(x_i) \Big] \, \mathcal{S}^{-1}(x_i) = \mathcal{S}(x) \prod_{i=1}^{N} \Big[1 + ig\Delta x_i^\mu A_\mu(x_i) \Big]$$

$$\cdot \, \mathcal{S}^{-1}(y) = \mathcal{S}(x) \, W_c(x,y) \, \mathcal{S}^{-1}(y).$$

$$\Rightarrow \boxed{W_c(x,y) \longrightarrow \mathcal{S}(x) W_c(x,y) \, \mathcal{S}^{-1}(y)}$$

Def. Wilson loop:

$\mathrm{tr}[W_c(x,x)]$ is called a <u>Wilson loop</u>.



(K. Wilson, '74?)

Under gauge transformation

$$\mathrm{tr}[W_c(x,x)] \to \mathrm{tr}[\mathcal{S}(x) W_c(x,x) \mathcal{S}^{-1}(x)] = \mathrm{tr}[W_c(x,x)]$$

invariant! Wilson loop is gauge-invariant!

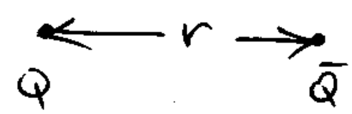
Uses:

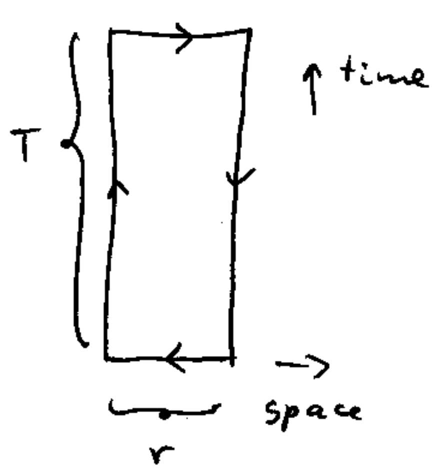
⇒ Wilson line represents quark propagator when one can neglect recoil. This works in high energy scattering and for static heavy quarks.

⇒ Wilson lines form links which can be used to define QCD action on the lattice for numerical simulations.

Heavy Quark Potential:

Suppose one wants to find heavy $Q\bar{Q}$ potential in QCD. How does one define the potential $V(r)$ in a gauge-invariant way?





Take a Wilson loop defined as shown.

$$\langle W \rangle \Big|_{T\to\infty} \simeq e^{-iTV(r)}$$

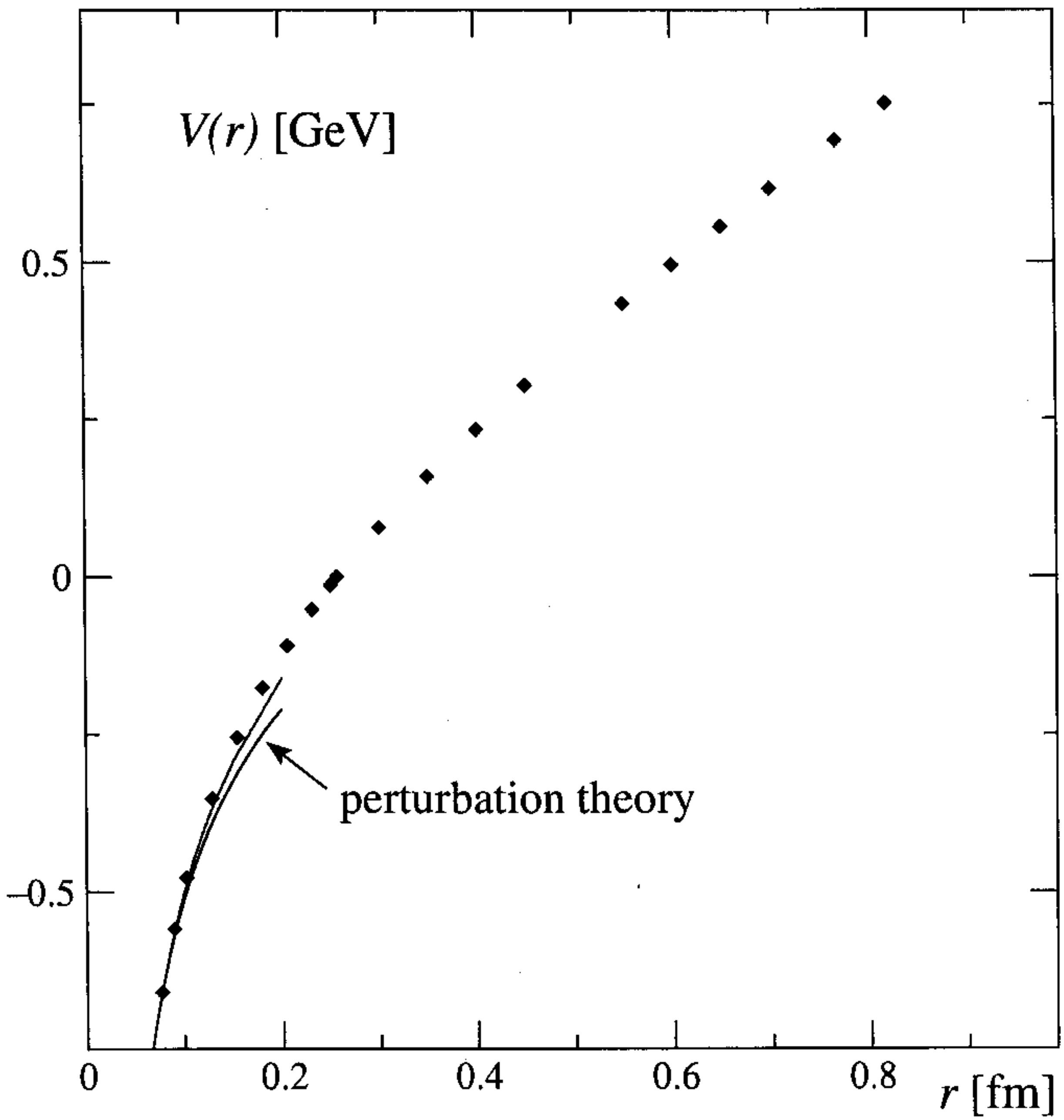
neglect interaction with gauge links.

$$\boxed{V(r) = \lim_{T\to\infty} \left[\frac{i}{T} \ln \langle W \rangle \right]}$$

~ can calculate numerically on the lattice

Lattice QCD: data points

perturbative QCD: solid lines + band.



$r \sim$ the only scale in $V(r) \Rightarrow \alpha_s = \alpha_s\left(\frac{1}{r^2}\right)$.

$\Rightarrow$ if $r \ll \frac{1}{\Lambda_{QCD}} \Rightarrow \alpha_s \ll 1 \Rightarrow$ perturbative QCD

applies $\Rightarrow$  ~ just like in QED,

get $V(r) \propto -\frac{\alpha}{r}$,

except that now one has a color factor:

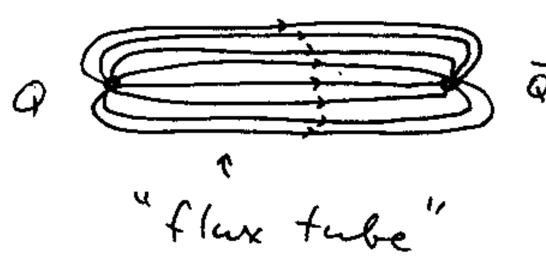
$$\frac{\mathrm{tr}\,(T^a T^a)}{\mathrm{tr}\,\mathbb{1}} = \frac{C_F N_c}{N_c} = C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$$

$\uparrow$ no interaction

$$\Rightarrow \boxed{V(r)\Big|_{r \ll \frac{1}{\Lambda_{QCD}}} = -\frac{4}{3}\frac{\alpha_s}{r}}.$$

$\Rightarrow$ if $r \gtrsim \frac{1}{\Lambda_{QCD}} \Rightarrow \alpha_s \sim 1 \Rightarrow$ perturbative approach breaks down $\Rightarrow$ non-perturbative physics.

<u>Qualitative picture:</u>



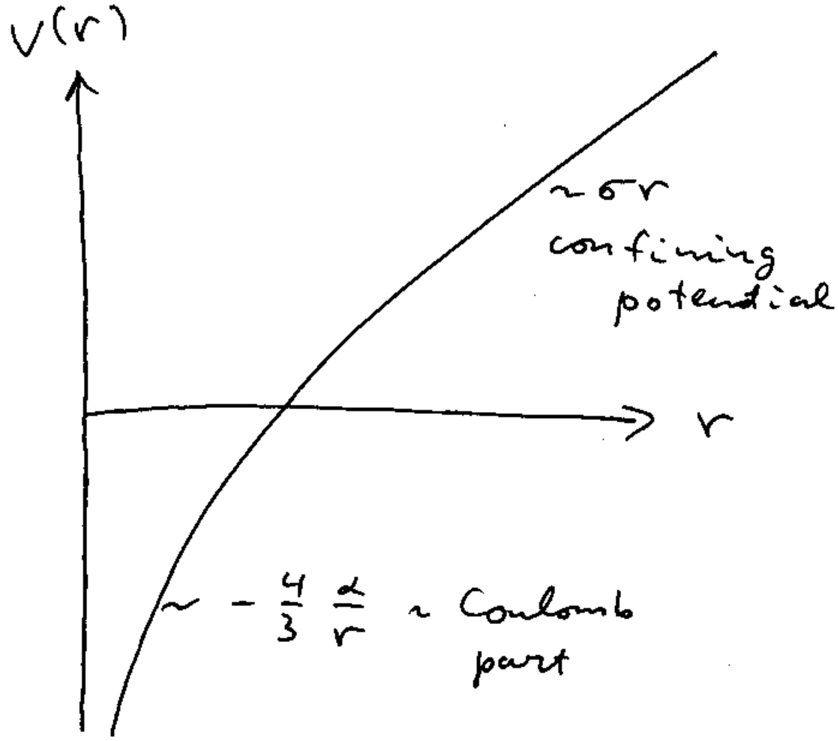
Constant force in flux tube

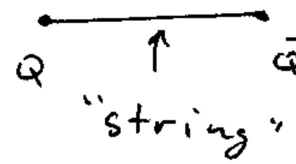
$\Rightarrow V(r) \sim F \cdot r \Rightarrow \boxed{V(r)\Big|_{r\Lambda_{QCD} \gtrsim 1} \simeq \sigma r}$

$\uparrow$ force

σ has dimension of Λ^2, $\sigma \sim \Lambda_{QCD}^2$

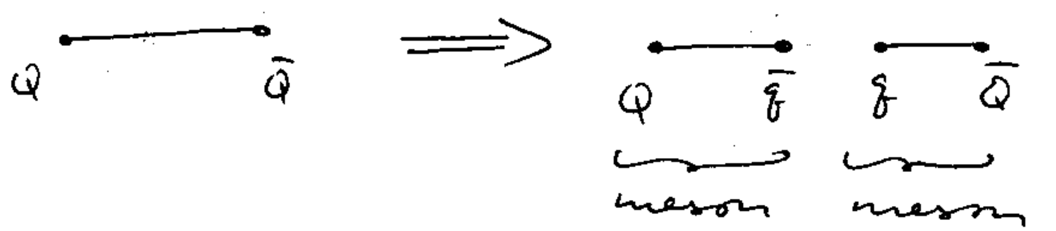
(string tension) $\sigma \simeq 1 \frac{GeV}{fm} \simeq 0.5\, GeV^2$.





$V(r)\Big|_{r \gtrsim \frac{1}{\Lambda_{QCD}}} \sim \sigma r$ ~ confines quarks, the phenomenon of quark confinement, not understood analytically, only see it on lattice (see attached hadout).

at very large r "string" breaks:



$q\bar{q}$ gets pulled out of the vacuum!