Expand $\Sigma_2(p)$ around $d=4$: $\varepsilon = 4-d \Rightarrow d = 4-\varepsilon$

$$\Rightarrow \Sigma_2(p) = \frac{e^2}{(4\pi)^{2-\frac{\varepsilon}{2}}} \int_0^1 dx \, \frac{\Gamma(\frac{\varepsilon}{2})\,[(-2+\varepsilon)x\not p + (4-\varepsilon)m]}{[(1-x)m^2 - x(1-x)p^2]^{\varepsilon/2}}$$

$$= \frac{\alpha}{4\pi}\int_0^1 dx\,[(-2+\varepsilon)x\not p + (4-\varepsilon)m]\cdot\left[\frac{2}{\varepsilon} - \gamma + \dots\right]\left[1+\frac{\varepsilon}{2}\ln 4\pi + \dots\right]$$

$$\cdot\left[1 - \frac{\varepsilon}{2}\ln[(1-x)m^2 - x(1-x)p^2] + \dots\right] = \frac{\alpha}{4\pi}\int_0^1 dx\,[\underbrace{(\varepsilon-2)}_{-2(1-\frac{\varepsilon}{2})}x\not p +$$

$$+ \overbrace{(4-\varepsilon)}^{=4(1-\frac{\varepsilon}{4})}m]\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \ln((1-x)m^2 - x(1-x)p^2) + \mathcal{O}(\varepsilon)\right]$$

$$\boxed{\begin{aligned} = \frac{\alpha}{4\pi}\int_0^1 dx\,\Big\{ & -2x\not p\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - 1 - \ln((1-x)m^2 - x(1-x)p^2)\right] \\ & + 4m\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \frac{1}{2} - \ln((1-x)m^2 - x(1-x)p^2)\right]\Big\} = \Sigma_2(p)\end{aligned}}$$

Now, $\Sigma_{ren}(p) = \Sigma_2(p) - \not p\,\delta_2 + \delta_m \Rightarrow$

$$\Rightarrow \boxed{\begin{aligned}\Sigma_{ren}(p) = \not p\Big\{ & -\delta_2 - \frac{\alpha}{2\pi}\int_0^1 dx\cdot x\cdot\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - 1 - \right. \\ & \left. - \ln((1-x)m^2 - x(1-x)p^2)\right\} + \delta_m + m\frac{\alpha}{\pi}\int_0^1 dx\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma \right. \\ & \left. - \frac{1}{2} - \ln((1-x)m^2 - x(1-x)p^2)\right]\end{aligned}}$$

$\Rightarrow$ any δ_2, δ_m that cancel $\frac{1}{\varepsilon}$ divergences are good.

Minimal Subtraction (MS) scheme: pick δ_2, δ_m to cancel $\frac{1}{\varepsilon}$ terms only:

$$\boxed{\delta_2^{MS} = -\frac{\alpha}{4\pi}\left[\frac{2}{\varepsilon}\right] = -\frac{\alpha}{2\pi}\frac{1}{\varepsilon}. \qquad \delta_m^{MS} = -m\,\frac{2\alpha}{\pi}\cdot\frac{1}{\varepsilon}}$$

Modified Minimal Subtraction ($\overline{MS}$) scheme:

cancel $\frac{2}{\varepsilon} - \gamma + \ln 4\pi \Rightarrow$ (as they always come together)

$$\boxed{\begin{aligned}\delta_2^{\overline{MS}} &= -\frac{\alpha}{4\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma\right]\\ \delta_m^{\overline{MS}} &= -m\frac{\alpha}{\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma\right]\end{aligned}}$$

On-Shell scheme: $\Sigma_{ren}(p)\Big|_{\substack{p^2=m^2\\ \not p = m}} = 0$

$$\Rightarrow m\left\{-\delta_2 - \frac{\alpha}{2\pi}\int_0^1 dx\cdot x\cdot\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - 1 - 2\ln((1-x)m)\right]\right\}$$

$$+\delta_m + m\frac{\alpha}{\pi}\int_0^1 dx\left\{\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \frac{1}{2} - 2\ln((1-x)m)\right\} = 0$$

$$\Rightarrow -\delta_2 m - m\frac{\alpha}{2\pi}\frac{1}{2}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - 1 - 2\ln m + 3\right] + \delta_m$$

$$+ m\frac{\alpha}{\pi}\left\{\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \frac{1}{2} - 2\ln m + 2\right\} = 0$$

$$\Rightarrow \boxed{m\,\delta_2 - \delta_m = m\,\frac{\alpha}{\pi}\left[\frac{3}{2}\frac{1}{\varepsilon} + \frac{3}{4}\ln 4\pi - \frac{3}{4}\gamma + 1 - \frac{3}{2}\ln m\right]}$$

$$\left.\frac{\partial \Sigma_{ren}(p)}{\partial \not p}\right|_{\not p = m} = \left.\frac{\partial \Sigma_2}{\partial \not p}\right|_{\not p = m} - \delta_2 = 0$$

$$\Rightarrow -\ \delta_2 - \frac{\alpha}{2\pi}\int_0^1 dx . x . \left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - 1 - 2\ln((1-x)m)\right]$$

$$+ \frac{\alpha}{2\pi}\cancel{m}\int_0^1 dx . x . \frac{-2x\cancel{(1-x)}\cancel{m}}{(1-x)^{\cancel{2}}\cancel{m^2}} - \frac{\alpha}{\pi}\cancel{m}\int_0^1 dx . \frac{-2x(1-x)\cancel{m}}{(1-x)^2\cancel{m^2}} = 0$$

$$-\ \delta_2 - \frac{\alpha}{2\pi}\frac{1}{2}\left[\frac{2}{\varepsilon} + \ln(4\pi) - \gamma - 1 - 2\ln m + 3\right] - \frac{\alpha}{\pi}\underbrace{\int_0^1 dx . \frac{x^2 {}^{+1-1}}{1-x}}_{\underbrace{-\int_0^1 dx\,(x+1)}_{-\frac{3}{2}} + \int_0^1 dx \frac{1}{1-x}} +$$

$$+ 2\frac{\alpha}{\pi}\left[-1 + \int_0^1 \frac{dx}{1-x}\right]$$

$$-\ \delta_2 - \frac{\alpha}{4\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma + 2 - 2\ln m\right] - \frac{\alpha}{\pi}\left[\frac{1}{2} - \int_0^1 \frac{dx}{1-x}\right] = 0$$

$$\boxed{\delta_2^{\text{on-shell}} = -\frac{\alpha}{4\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma + 4 - 2\ln m - 4\int_0^1 \frac{dx}{1-x}\right]}$$

$$\delta_m = m\,\delta_2 - m\,\frac{\alpha}{\pi}\left[\frac{3}{4}\left(\frac{2}{\varepsilon} + \ln 4\pi - \gamma\right) + 1 - \frac{3}{2}\ln m\right]$$

$$\Rightarrow \boxed{\delta_m^{\text{on-shell}} = -m\,\frac{\alpha}{\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma + 2 - 2\ln m - \int_0^1 \frac{dx}{1-x}\right]}$$

$\Rightarrow$ Note that δ_2, δ_m in different schemes differ only by finite (or IR-divergent) pieces!

$$\Sigma_{ren}^{on\text{-}shell}(p) = \not{p}\left\{-\frac{\alpha}{\pi}\int_0^1 \frac{dx}{1-x} + \frac{\alpha}{2\pi}\int_0^1 dx\cdot x\cdot\left[\ln\left[\frac{(1-x)m^2 - x(1-x)p^2}{m^2}\right]+5\right]\right\}$$

$$+ m\,\frac{\alpha}{\pi}\int_0^1 dx\left\{\ln\left[\frac{m^2}{(1-x)m^2 - x(1-x)p^2}\right] - \frac{5}{2}\right\} + m\,\frac{\alpha}{\pi}\int_0^1 \frac{dx}{1-x}$$

$$\Rightarrow \boxed{\begin{aligned}\Sigma_{ren}^{on\text{-}shell}(p) &= \not{p}\,\frac{\alpha_{EM}}{2\pi}\int_0^1 dx\cdot x\cdot\left\{\ln\left[\frac{(1-x)m^2 - x(1-x)p^2}{m^2}\right]+5\right\} \\ &- m\,\frac{\alpha_{EM}}{\pi}\int_0^1 dx\left\{\ln\left[\frac{(1-x)m^2 - x(1-x)p^2}{m^2}\right]+\frac{5}{2}\right\} - (\not{p}-m)\cdot \\ &\cdot\frac{\alpha_{EM}}{\pi}\cdot\int_0^1 \frac{dx}{1-x}\end{aligned}}$$

~ no more UV divergences ⇒ renormalized 2-point function!

$$\Pi_{ren}^{on\text{-}shell}(q^2) = \Pi_2(q^2) - \Pi_2(0) = \frac{\alpha}{3\pi}\ln\left(\frac{-q^2}{m^2 e^{5/3}}\right)$$

$$\Rightarrow \boxed{\Pi_{ren}^{on\text{-}shell}(q^2) = \frac{\alpha_{EM}}{3\pi}\ln\left[\frac{-q^2}{m^2 e^{5/3}}\right]}$$

also UV-finite!