Renormalization of QED:

Start with the QED Lagrangian written in terms of bare fields, mass & coupling:

$$\mathcal{L}_{QED} = \bar{\psi}_0 [i\not{\partial} - m_0]\psi_0 - \frac{1}{4} F^0_{\mu\nu} F^{0\mu\nu} - e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A^0_\mu$$

where

ψ_0 = bare electron field

A_μ^0 = bare photon field, $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$

e_0 = bare coupling

m_0 = bare electron mass

Rescale the fields: $\psi_0 = \sqrt{Z_2}\,\psi$, $A_\mu^0 = \sqrt{Z_3}\,A_\mu$

with ψ, A_μ = renormalized fields

The Lagrangian becomes

$$\mathcal{L}_{QED} = Z_2 \bar{\psi}[i\not{\partial} - m_0]\psi - \frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} - e_0 Z_2 Z_3^{1/2} \cdot \bar{\psi}\gamma^\mu \psi A_\mu$$

The factors Z_2, Z_3 may (and do) contain infinities. To regulate them work in d-dimensions.

(dimensional regularization)

The action $S = \int d^d x \, \mathcal{L}$ is always dimensionless

$\Rightarrow [\mathcal{L}] = M^d$. Factors Z_2, Z_3 are dimensionless

$\Rightarrow$ taking $-m_0 Z_2 \bar{\psi}\psi$ term $\sim M^d$ we see that

$$[\psi] = M^{\frac{d-1}{2}}$$

taking $-\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu}$ term $\sim M^d$ we get

$$[A_\mu] = M^{\frac{d-2}{2}}$$

(Cross-check: for $d = 4$ get $[\psi] = M^{3/2}$, $[A_\mu] = M$ as expected.)

As $e_0 \bar{\psi} \gamma^\mu \psi A_\mu \sim M^d \Rightarrow e_0 M^{d-1} M^{\frac{d-2}{2}} \sim M^d$

$$\Rightarrow [e_0] = M^{\frac{4-d}{2}} = M^{\varepsilon/2}$$ where $\boxed{d = 4 - \varepsilon}$.

The coupling is dimensionful! We can show its dimensionality explicitly by

(a) defining the renormalized charge e and a factor Z_1 by

$$\boxed{e_0 Z_2 Z_3^{1/2} \equiv e Z_1}$$

(b) and by rewriting

$$\boxed{e = e_\mu \mu^{\varepsilon/2}}$$

where μ is an arbitrary momentum scale.

μ = renormalization scale (arbitrary)

e_μ = dimensionless(!) renormalized coupling

The QED Lagrangian is

$$\mathcal{L}_{QED} = Z_2 \bar{\Psi}[i\not{\partial} - m_0]\Psi - \frac{1}{4} Z_3 F_{\mu\nu}F^{\mu\nu} - e_\mu \mu^{\varepsilon/2} Z_1 \bar{\Psi}\gamma^\mu \Psi A_\mu .$$

We want to separate the "old" Lagrangian without the Z's from the rest. We write

$$\mathcal{L}_{QED} = \bar{\Psi}[i\not{\partial} - m]\Psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - e_\mu \mu^{\varepsilon/2} \bar{\Psi}\gamma^\mu \Psi A_\mu$$

$$+ \bar{\Psi}[(Z_2 - 1) i\not{\partial} - Z_2 m_0 + m]\Psi - \frac{1}{4}(Z_3 - 1) F_{\mu\nu}F^{\mu\nu}$$

$$- (Z_1 - 1) e_\mu \mu^{\varepsilon/2} \bar{\Psi}\gamma^\mu \Psi A_\mu .$$

Here m = renormalized mass (can be the physical e^- mass, or it could be off from it by a finite amount).

Define

$$\boxed{\begin{aligned} &\delta_3 \equiv Z_3 - 1, \quad \delta_2 \equiv Z_2 - 1, \quad \delta_m \equiv Z_2 m_0 - m, \\ &\delta_1 = Z_1 - 1 = \frac{e_0}{e} Z_2 Z_3^{1/2} - 1 = \frac{e_0}{e_\mu \mu^{\varepsilon/2}} Z_2 Z_3^{1/2} - 1 \end{aligned}}$$

These δ's are the counterterm coefficients (sometimes also called counterterms).

The Lagrangian of QED becomes

$$\boxed{\begin{aligned}\mathcal{L}_{QED} &= \bar{\Psi}\left[i\not{\partial}-m\right]\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e_\mu \mu^{\varepsilon/2}\bar{\Psi}\gamma^\mu\Psi A_\mu \\ &\quad - \frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left[i\delta_2\not{\partial} - \delta_m\right]\Psi - e_\mu\mu^{\varepsilon/2}\delta_1\bar{\Psi}\gamma^\mu\Psi A_\mu\end{aligned}}$$

$\Rightarrow$ Note that this is <u>the same</u> QED Lagrangian, written in terms of renormalized fields and couplings.

$\Rightarrow$ first line = renormalized Lagrangian

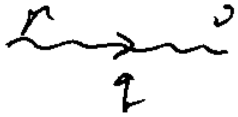
second line = "counterterms"

(Sometimes people say that they "added" counterterms: this is not true, counterterms emerge from the $\mathcal{L}$ written originally in terms bare fields after field redefinitions.)

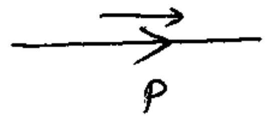
$\Rightarrow$ as we will shortly see, all δ's are at least order-α_μ $\left(\text{with } \alpha_\mu = \frac{e_\mu^2}{4\pi}\right)$, such that the counterterms are the new interaction terms, in addition to $e\bar{\Psi}\gamma^\mu\Psi A_\mu$. Hence

$$\boxed{\begin{aligned}\mathcal{L}_{free} &= \bar{\Psi}\left[i\not{\partial}-m\right]\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ \mathcal{L}_{int} &= -e_\mu\mu^{\varepsilon/2}\bar{\Psi}\gamma^\mu\Psi A_\mu + \bar{\Psi}\left[i\delta_2\not{\partial}-\delta_m\right]\Psi - \frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} \\ &\quad - e_\mu\mu^{\varepsilon/2}\delta_1\bar{\Psi}\gamma^\mu\Psi A_\mu .\end{aligned}}$$

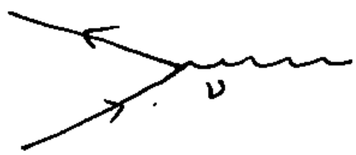
Feynman rules for the renormalized $\mathcal{L}_{QED}$ are:



$\dfrac{-i g_{\mu\nu}}{q^2+i\varepsilon}$;



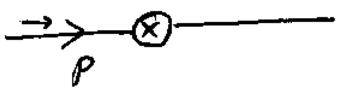
$= \dfrac{i}{\not{p}-m}$



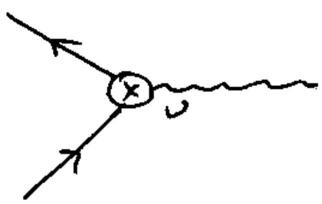
$-i e_r \mu^{\varepsilon/2} \gamma^\nu = -ie\gamma^\nu$

same as before

$-i\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]\delta_3$



$i\left(\not{p}\,\delta_2 - \delta_m\right)$



$-ie\,\delta_1\gamma^\nu = -i e_r \mu^{\varepsilon/2}\delta_1\gamma^\nu$

new vertices (counter-terms)

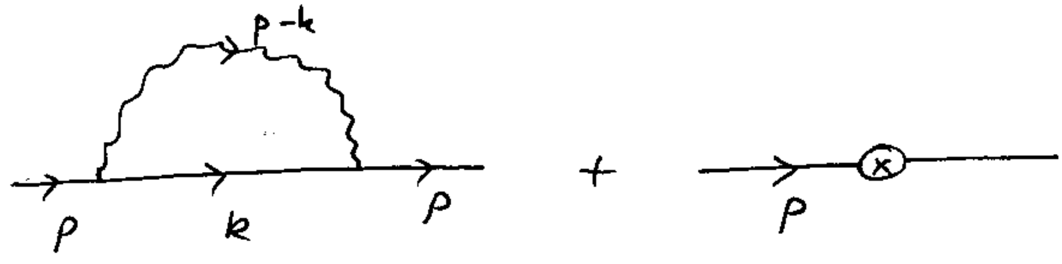
$\Rightarrow$ Counterterm coefficients $\delta_1, \delta_2, \delta_3, \delta_m$ will be (partially) fixed below by requiring that they cancel UV divergences arising from loop diagrams.

$\Rightarrow$ Electron mass m and charge e are fixed from the data. QED does not predict their values ($m_e = 511$ keV , $\alpha_{EM} = \dfrac{e^2}{4\pi} = \dfrac{1}{137}$).

These are external parameters in the theory.

One-loop structure of QED.

To fix the counterterms, let us revisit the divergences: start with the electron self-energy. There are 2 diagrams now contributing at $O(\alpha_\mu)$:



$$-i\,\Sigma_2^{ren}(p) = -i\,\Sigma_2(p) + i\,(\not p\,\delta_2 - \delta_m)$$

↖ renormalized self-energy

$$\Sigma_2^{ren}(p) = \Sigma_2(p) - \not p\,\delta_2 + \delta_m$$

where $\Sigma_2(p)$ is the same as before:

$$\Sigma_2(p) = -i\, e_\mu^2 \mu^\varepsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p-k)^2 + i\varepsilon}\, \gamma^\rho \frac{1}{\not k - m} \gamma_\rho \, .$$

Above we calculated Σ_2 using Pauli-Villars regularization. Let's do it again using dimensional regularization.

$\Rightarrow$ In d-dimensions we still have

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \Rightarrow \gamma^\mu \gamma_\mu = \delta^\mu{}_\mu = d$$

$$\gamma^\rho \gamma^\nu \gamma_\rho = \underbrace{\{\gamma^\rho, \gamma^\nu\}}_{2g^{\nu\rho}} \gamma_\rho - \gamma^\nu \underbrace{\gamma^\rho \gamma_\rho}_{d} = 2\gamma^\nu - d\gamma^\nu$$
$$= (2-d)\gamma^\nu$$

$\Rightarrow$ we have

$$\boxed{\begin{aligned} \gamma_\mu \gamma^\mu &= d \\ \gamma^\rho \gamma^\nu \gamma_\rho &= (2-d)\gamma^\nu \end{aligned}}$$

(again, these work fine for $d=4$.)

$$\Rightarrow \quad \Sigma_2(p) = -i e_\mu^2 \mu^\varepsilon \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)\not{k} + d\, m}{[(p-k)^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]}.$$

$\Rightarrow$ Introduce Feynman parameters, do the Wick rotation and integrate over 4-momentum to obtain (cf. Eq. (10.41) in Peskin & Schroeder):

$$\Sigma_2(p) = \frac{e_\mu^2 \mu^\varepsilon}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{\Gamma\left(2-\frac{d}{2}\right)\left[(2-d)x\not{p} + d\, m\right]}{\left[(1-x)m^2 - x(1-x)p^2\right]^{2-d/2}}.$$

Expand $\Sigma_2(p)$ around $d=4$: as usual $\varepsilon = 4-d$

$$\Rightarrow \Sigma_2(p) = \frac{e_\mu^2 \mu^\varepsilon}{(4\pi)^{2-\varepsilon/2}} \int_0^1 dx \, \frac{\Gamma(\varepsilon/2)\left[(-2+\varepsilon)x\not{p} + (4-\varepsilon)m\right]}{\left[(1-x)m^2 - x(1-x)p^2\right]^{\varepsilon/2}}$$

$$\Sigma_2(p) = \frac{\alpha_r \mu^\varepsilon}{4\pi} \int_0^1 dx \left[(-2+\varepsilon)x\not{p} + (4-\varepsilon)m\right]\left[\frac{2}{\varepsilon} - \gamma + \ldots\right]$$

$$\left[1 + \frac{\varepsilon}{2}\ln 4\pi + \ldots\right]\left[1 - \frac{\varepsilon}{2}\ln\left[(1-x)m^2 - x(1-x)p^2\right] + O(\varepsilon^2)\right] =$$

$$= \frac{\alpha_r \mu^\varepsilon}{4\pi} \int_0^1 dx \left[(-2+\varepsilon)x\not{p} + (4-\varepsilon)m\right]\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma\right.$$

$$\left. - \ln\left[(1-x)m^2 - x(1-x)p^2\right] + O(\varepsilon)\right].$$

Finally, rewriting $\mu^\varepsilon = 1 + \varepsilon \ln\mu + O(\varepsilon^2)$ we get

$$\Sigma_2(p) = \frac{\alpha_r}{4\pi} \int_0^1 dx \left\{ -2x\not{p}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - 1 - \ln\frac{(1-x)m^2 - x(1-x)p^2}{\mu^2}\right]\right.$$

$$\left. + 4m\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \frac{1}{2} - \ln\frac{(1-x)m^2 - x(1-x)p^2}{\mu^2}\right]\right\}$$

as $\Sigma_2^{ren}(p) = \Sigma_2(p) - \not{p}\,\delta_2 + \delta_m \Rightarrow$

$$\Sigma_2^{ren}(p) = \not{p}\left\{-\delta_2 - \frac{\alpha_r}{2\pi}\int_0^1 dx \cdot x \cdot \left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - 1 - \ln\frac{(1-x)m^2 - x(1-x)p^2}{\mu^2}\right]\right\}$$

$$+ \delta_m + m\frac{\alpha_r}{\pi}\int_0^1 dx \left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \frac{1}{2} - \ln\frac{(1-x)m^2 - x(1-x)p^2}{\mu^2}\right].$$

$\Rightarrow$ δ_2 & δ_m can be fixed by requiring that they cancel $\frac{2}{\varepsilon}$ divergences in the $\not{p}$ and $\mathbb{1}_{4\times4}$ structures in Σ_2^{ren}, making Σ_2^{ren} finite:

we see that

$$\boxed{\begin{aligned} \delta_2 &= -\frac{\alpha_\mu}{2\pi}\frac{1}{\varepsilon} + \text{finite} \\ \delta_m &= -m\,\frac{2\alpha_\mu}{\pi}\frac{1}{\varepsilon} + \text{finite} \end{aligned}}$$

where the finite parts are arbitrary. They are fixed according to conventions (aka schemes). Popular schemes are:

Minimal Subtraction (MS) scheme: pick δ_2, δ_m to cancel divergences only, no finite terms:

$$\boxed{\begin{aligned} \delta_2^{MS} &= -\frac{\alpha_\mu}{4\pi}\frac{2}{\varepsilon} = -\frac{\alpha_\mu}{2\pi}\frac{1}{\varepsilon} \\ \delta_m^{MS} &= -m\,\frac{2\alpha_\mu}{\pi}\frac{1}{\varepsilon} \end{aligned}}$$

$\Rightarrow$ one can easily write down Σ_2^{ren} in MS scheme

Modified Minimal Subtraction ($\overline{MS}$) scheme:

cancel $\frac{2}{\varepsilon} + \ln 4\pi - \gamma$ terms (they always come together):

$$\boxed{\begin{aligned} \delta_2^{\overline{MS}} &= -\frac{\alpha_\mu}{4\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma\right] \\ \delta_m^{\overline{MS}} &= -m\,\frac{\alpha_\mu}{\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma\right] \end{aligned}}$$

$$\boxed{\begin{aligned}\Sigma_2^{\overline{MS}}(p) &= \not{p}\,\frac{\alpha_\mu}{2\pi}\int_0^1 dx\cdot x\cdot\left[1+\ln\frac{(1-x)m^2-x(1-x)p^2}{\mu^2}\right]\\ &\quad - m\,\frac{\alpha_\mu}{\pi}\int_0^1 dx\left[\frac{1}{2}+\ln\frac{(1-x)m^2-x(1-x)p^2}{\mu^2}\right]\end{aligned}}$$

$\Rightarrow$ no $\frac{1}{\varepsilon}$ divergences in $\Sigma_2^{\overline{MS}}$.

$\Rightarrow$ there is still μ^2-dependence (address later)

<u>On-Shell (OS) scheme:</u>

(i) Want the renormalized electron propagator to be

$$= \frac{i}{\not{p}-m} + \begin{pmatrix}\text{terms regular}\\ \text{at } p^2=m^2\end{pmatrix}$$

with m the physical electron mass

$$\Rightarrow \quad S'(p) = \frac{i}{\not{p}-m-\Sigma_{ren}(p)} = \frac{i}{\not{p}-m} + \begin{pmatrix}\text{regular}\\ \text{terms}\end{pmatrix}$$

$$\Rightarrow \quad \boxed{\begin{aligned}\Sigma(p)\Big|_{\not{p}=m} &= 0\\ \frac{\partial\Sigma}{\partial\not{p}}\Big|_{\not{p}=m} &= 0\end{aligned}} \quad \left(\text{as } \frac{1}{Z_2} = 1-\frac{\partial\Sigma}{\partial\not{p}}\Big|_{\not{p}=m}\right) \quad \Rightarrow \text{give } \delta_2, \delta_m$$

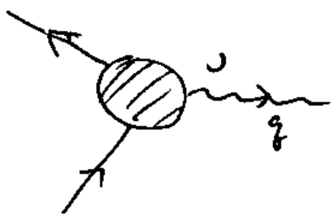
(ii) Want the renormalized photon propagator to be

$$= \frac{-i g_{\mu\nu}}{q^2+i\varepsilon} \quad \Rightarrow \text{as the dressed photon}$$

propagator is $\dfrac{-i g_{\mu\nu}}{q^2[1-\Pi_{ren}(q^2)]}$ $\Rightarrow$

get $\Pi_{ren}(q^2=0) = 0$.

(iii) Want the renormalized electron charge to be e $\left(\alpha_{EM} = \dfrac{e^2}{4\pi} = \dfrac{1}{137}\right)$



$$= -ie\gamma^\nu = -ie_r\mu^{\varepsilon/2}\gamma^\nu$$

$$\Rightarrow \boxed{\Gamma^\nu(q=0) = \gamma^\nu}.$$

In the OS scheme we get:

$$0 = \Sigma_2^{ren}\left(\begin{matrix}p^2=m^2\\ \not p = m\end{matrix}\right) = m\left\{-\delta_2 - \frac{\alpha_\mu}{2\pi}\int_0^1 dx\cdot x\cdot\left[\frac{2}{\varepsilon}+\ln 4\pi - \gamma - 1\right.\right.$$

$$\left.\left.-2\ln\frac{(1-x)m}{\mu}\right]\right\} + \delta_m + m\,\frac{\alpha_\mu}{\pi}\int_0^1 dx\left[\frac{2}{\varepsilon}+\ln 4\pi-\gamma-\frac{1}{2}\right.$$

$$\left.-2\ln\frac{(1-x)m}{\mu}\right] \Rightarrow \text{simplifying this we get}$$

$$-m\delta_2 - m\,\frac{\alpha_\mu}{4\pi}\left[\frac{2}{\varepsilon}+\ln 4\pi-\gamma-1-\ln\frac{m^2}{\mu^2}+3\right]+\delta_m$$

$$+m\,\frac{\alpha_\mu}{\pi}\left[\frac{2}{\varepsilon}+\ln 4\pi-\gamma-\frac{1}{2}-\ln\frac{m^2}{\mu^2}+2\right] = 0$$

$$\Rightarrow \boxed{m\delta_2^{os} - \delta_m^{os} = m\frac{3\alpha_\mu}{4\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma + \frac{4}{3} - \ln\frac{m^2}{\mu^2}\right]}$$

$$0 = \left.\frac{\partial \Sigma_2^{ren}(p)}{\partial \not{p}}\right|_{\not{p}=m} = \left.\frac{\partial \Sigma_2}{\partial \not{p}}\right|_{\not{p}=m} - \delta_2^{os} = 0$$

$$\Rightarrow \delta_2^{os} = \left.\frac{\partial \Sigma_2}{\partial \not{p}}\right|_{\not{p}=m} = -\frac{\alpha_\mu}{2\pi}\int_0^1 dx\cdot x\cdot\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - 1 - 2\ln\frac{(1-x)m}{\mu}\right]$$

$$+\frac{\alpha_\mu}{2\pi} m\int_0^1 dx\cdot x\cdot\frac{-2x\cancel{(1-x)}m}{(1-x)^{\cancel{2}}m^{\cancel{2}}} - \frac{\alpha_\mu}{\pi} m\int_0^1 dx\,\frac{-2x\cancel{(1-x)m}}{(1-x)^2 m^{\cancel{2}}}$$

This leads to

$$\boxed{\delta_2^{os} = -\frac{\alpha_\mu}{4\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma + 4 - \ln\frac{m^2}{\mu^2} - 4\int_0^1\frac{dx}{1-x}\right]}$$

$$\delta_m^{os} = m\delta_2^{os} - m\frac{3\alpha_\mu}{4\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma + \frac{4}{3} - \ln\frac{m^2}{\mu^2}\right]$$

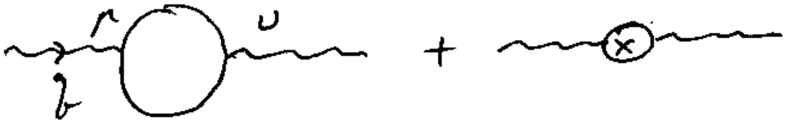
$$\Rightarrow \boxed{\delta_m^{os} = -m\frac{\alpha_\mu}{\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma + 2 - \ln\frac{m^2}{\mu^2} - \int_0^1\frac{dx}{1-x}\right]}$$

Plugging this back in we get

$$\boxed{\begin{aligned}\Sigma_2^{os}(p) &= \not{p}\frac{\alpha_\mu}{2\pi}\int_0^1 dx\cdot x\cdot\left\{\ln\left[\frac{(1-x)m^2 - x(1-x)p^2}{m^2}\right] + 5\right\}\\ &- m\frac{\alpha_\mu}{\pi}\int_0^1 dx\left\{\ln\left[\frac{(1-x)m^2 - x(1-x)p^2}{m^2}\right] + \frac{5}{2}\right\} - (\not{p}-m)\frac{\alpha_\mu}{\pi}\int_0^1\frac{dx}{1-x}.\end{aligned}}$$

(Note: no explicit μ-dependence, only in α_μ.)

Photon self-energy also has two contributing diagrams at one-loop:



$$\Rightarrow \quad i\,\Pi^{\mu\nu}_{2,\text{ren}}(q) = i\,\Pi^{\mu\nu}_{2}(q) - i\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]\delta_3$$

$$\Rightarrow \quad \Pi_2^{\text{ren}}(q^2) = \Pi_2(q^2) - \delta_3$$

Above we found $\Pi_2(q^2)$ using dim. reg.:

$$\Pi_2(q^2) = -\frac{2\alpha_\mu \mu^\varepsilon}{\pi}\int_0^1 dx \cdot x \cdot (1-x)\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \ln\left(m^2 - x(1-x)q^2\right)\right]$$

$$\Rightarrow \quad \Pi_2^{\text{ren}}(q^2) = -\delta_3 - \frac{2\alpha_\mu}{\pi}\int_0^1 dx \cdot x \cdot (1-x)\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \ln\frac{m^2 - x(1-x)q^2}{\mu^2}\right]$$

δ_3 is fixed by requiring that it removes the $\frac{1}{\varepsilon}$-divergence in Π_2^{ren}. We get

$$\delta_3^{MS} = -\frac{\alpha_\mu}{3\pi}\,\frac{2}{\varepsilon} \qquad\qquad \delta_3^{\overline{MS}} = -\frac{\alpha_\mu}{3\pi}\left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right]$$

$$\Pi_2^{\overline{MS}}(q^2) = \frac{2\alpha_\mu}{\pi}\int_0^1 dx \cdot x \cdot (1-x)\,\ln\left[\frac{m^2 - x(1-x)q^2}{\mu^2}\right]$$

$\Rightarrow$ $\Pi_2^{\overline{MS}}$ is finite!

On-shell scheme:

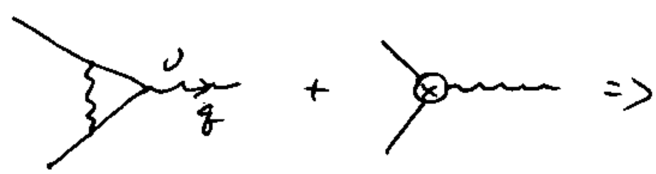
$$\Pi_2^{ren}(q^2=0) = 0 = -\delta_3^{os} - \frac{2\alpha_\mu}{\pi}\int_0^1 dx\cdot x\cdot(1-x)\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \ln\frac{m^2}{\mu^2}\right]$$

$$\Rightarrow \delta_3^{os} = -\frac{\alpha_\mu}{3\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \ln\frac{m^2}{\mu^2}\right]$$

$$\Pi_2^{os}(q^2) = \Pi_2(q^2) - \delta_3^{os} \Rightarrow$$

$$\boxed{\Pi_2^{os}(q^2) = +\frac{2\alpha_\mu}{\pi}\int_0^1 dx\cdot x\cdot(1-x)\ln\left[\frac{m^2 - x(1-x)q^2}{m^2}\right].}$$

- again, no explicit μ-dependence
- Π_2^{os} is also finite.

Vertex corrections: 

$$-ie\Gamma^{\nu}_{2,ren}(q) = -ie\left[\Gamma_2^{\nu}(q) + \delta_1\gamma^\nu\right]$$

$\Rightarrow$ $\Gamma^{\nu}_{2,ren}(q) = \Gamma_2^{\nu}(q) + \delta_1\gamma^\nu$ $\Rightarrow$ fix δ_1 by requiring $\Gamma^{\nu}_{2,ren}(q)$ to be finite.

Above we used Ward identity to show that

$$\Gamma_2^{\nu}(q\to 0) = \frac{1}{Z_2}\gamma^\nu = \frac{1}{1+\delta_2}\gamma^\nu = (1-\delta_2+\ldots)\gamma^\nu$$

$\Rightarrow \Gamma^{\nu}_{2,ren}(q \to 0) = \gamma^{\nu}\left[1 - \delta_2 + \delta_1 + O(\alpha_\mu^2)\right] = \text{finite}$

$\Rightarrow$ clearly in MS & $\overline{MS}$ $\quad \delta_2^{\overline{MS}} = \delta_1^{\overline{MS}}$. (divergences must cancel)

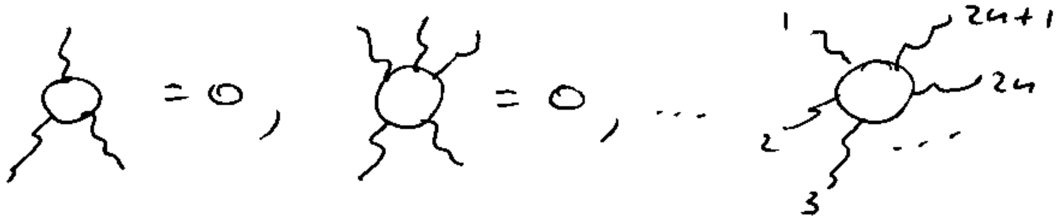
In the on-shell scheme require that

$\Gamma^{\nu}_{2,ren}(q=0) = \gamma^{\nu} \Rightarrow \boxed{\delta_1^{OS} = \delta_2^{OS}}$ as well!
(true to all orders in α_μ)

$\Rightarrow$ we fixed $\delta_1, \delta_2, \delta_3$ & δ_m in MS, $\overline{MS}$,

and OS schemes!

$\Rightarrow$ One can show that there are no other divergences in QED at one-loop order.



$= 0$ ($\forall$ odd # of legs)

Furry's theorem: $\langle 0 | T A_{\mu_1}(x_1) \ldots A_{\mu_{2n+1}}(x_{2n+1}) | 0 \rangle = \Gamma_{2n+1}$

$\Rightarrow$ under charge conjugation (see Peskin & Schroeder, ch 3, 6)

$\bar{\psi}\gamma^{\mu}\psi \xrightarrow{C} -\bar{\psi}\gamma^{\mu}\psi \Rightarrow A_\mu \xrightarrow{C} -A_\mu \Rightarrow$

$\Gamma_{2n+1} \xrightarrow{C} (-1)^{2n+1}\,\Gamma_{2n+1} = -\Gamma_{2n+1} \Rightarrow$

as $\mathcal{L}_{QED}$ is C-invariant $\Rightarrow \Gamma_{2n+1} \xrightarrow{C} \Gamma_{2n+1}$

$\Rightarrow \Gamma_{2n+1} = -\Gamma_{2n+1} \Rightarrow \boxed{\Gamma_{2n+1} = 0}$ Furry's theorem

$\sim \int^{\Lambda} \frac{d^4k}{k^4} \sim \ln \Lambda \Rightarrow$ in fact finite (can show).

In general can characterize the diagram by its <u>superficial degree of divergence</u>: $D = 4L - P_e - 2P_\gamma$

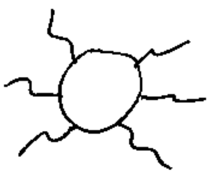
L = # loops (each loop gives d^4k)

P_e = # of electron propagators (each fermion prop. gives $1/k$)

P_γ = # —"— photon —"— (each gives $1/k^2$).

$\Rightarrow$ the diagram should diverge at most as Λ^D.

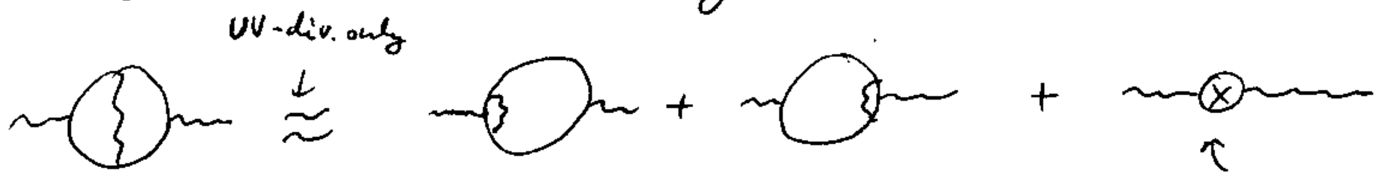
(if $D<0$ + subdiagrams $\Rightarrow$ convergent diagram Weinberg's th'm)



$\sim \Lambda^{4\cdot 1-6} \sim \Lambda^{-2} \sim \frac{1}{\Lambda^2} \Rightarrow$ finite

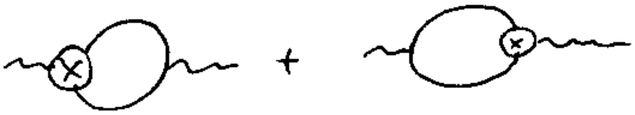
$L=1,\ P_e=6,\ P_\gamma=0$ (all other multi-leg 1-loops are finite too)

What about multi-loop graphs? One can show that UV divergences are removed by counterterms:

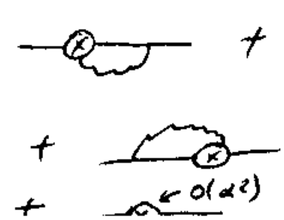


counterterm at $O(\alpha^2)$

$\Rightarrow$ 1st & 2nd terms are removed by



$\Rightarrow$ removed by



$\Rightarrow$ QED is renormalizable!

In general one can tell if the theory is renormalizable by dimension of the coupling constant: if $[\lambda]=M^n$

$\Rightarrow \lambda \sim M^n \Rightarrow$ each λ comes with $\frac{1}{p^n} \Rightarrow$ get $\left(\frac{M}{p}\right)^n$.

$\Rightarrow$ $n > 0$ $\Rightarrow$ higher orders have less UV divergences than lower orders

$n = 0$ $\Rightarrow$ higher order graphs are as divergent as lower order ones.

$n < 0$ $\Rightarrow$ higher order graphs are more divergent than LO ones.

$\Rightarrow$ $n > 0$: <u>super-renormalizable</u> theory. (e.g. φ^4 in 3-dim)

$n = 0$: <u>renormalizable</u> theory (φ^4 in 4-dim, QED)

$n < 0$: <u>non-renormalizable</u> theory (e.g. φ^6 in 4 dim.)

Running of QED Coupling Constant

We have renormalized QED. However, we have an unknown function α_μ, which we need to find. Start with

$$e_0 Z_2 Z_3^{1/2} = e_\mu \mu^{\varepsilon/2} Z_1 .$$

Since $\delta_1 = \delta_2 \Rightarrow Z_1 = 1 + \delta_1 = 1 + \delta_2 = Z_2 \Rightarrow$

$$\Rightarrow e_0 Z_3^{1/2} = e_\mu \mu^{\varepsilon/2} \Rightarrow \boxed{\alpha_0 Z_3 = \alpha_\mu \mu^\varepsilon}$$

where $\alpha_0 = \frac{e_0^2}{4\pi}$, $\alpha_\mu = \frac{e_\mu^2}{4\pi}$.

(Valid only in MS & $\overline{MS}$ schemes. For on-shell scheme, one has to take $\mu^2 \gg m^2$ and set the renormalization conditions at $q^2 = -\mu^2$, $p^2 = -\mu^2$ instead. See Weinberg, Sec. 18.2.)

$\Rightarrow$ as α_0 is bare coupling, it is μ-independent

$$\Rightarrow \quad \boxed{\mu^2 \frac{d\alpha_0}{d\mu^2} = 0}$$

Plugging $\alpha_0 = \frac{\alpha_\mu \mu^\varepsilon}{Z_3}$ in yields

In the OS scheme one can find the beta-function using Green functions. To use the above trick one has to set OS renormalization conditions at

$q^2 = -\mu^2$ with $\mu^2 \gg m^2$.

For instance, putting $\Pi^2_{ren}(q^2=-\mu^2) = 0$ for $m \ll \mu$

gives $-\delta_3^{OS} - \frac{2\alpha_\mu}{\pi} \int_0^1 dx.x.(1-x)\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \ln x(1-x)\right] = 0$

$$\Rightarrow \quad \delta_3^{OS} = -\frac{\alpha_\mu}{3\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi - \gamma - \frac{5}{3}\right]$$ (cf. p. 216)

Using this in $0 = \mu^2 \frac{d\alpha_0}{d\mu^2} = \mu^2 \frac{d}{d\mu^2}\left(\frac{\alpha_\mu \mu^\varepsilon}{1+\delta_3}\right)$

gives the standard one-loop QED β-function.

$$0 = \mu^2 \frac{d\alpha_0}{d\mu^2} = \mu^2 \frac{d}{d\mu^2}\left\{ (\mu^2)^{\varepsilon/2} \cdot \alpha_\mu \cdot \left[1 + \frac{\alpha_\mu}{3\pi} \cdot \frac{2}{\varepsilon} \right] \right\} =$$

$$= \mu^\varepsilon \frac{\varepsilon}{2} \alpha_\mu \left[1 + \frac{\alpha_\mu}{3\pi} \frac{2}{\varepsilon} \right] + \mu^2 \frac{d\alpha_\mu}{d\mu^2} \cdot \mu^\varepsilon \left[1 + \frac{\alpha_\mu}{3\pi} \frac{2}{\varepsilon} \right] + \mu^\varepsilon \alpha_\mu \cdot \frac{2}{\varepsilon} \frac{1}{3\pi} \cdot$$

$$\cdot \mu^2 \frac{d\alpha_\mu}{d\mu^2} \Rightarrow \mu^2 \frac{d\alpha_\mu}{d\mu^2} \left[1 + 2 \frac{\alpha_\mu}{3\pi} \frac{2}{\varepsilon} \right] = -\frac{\varepsilon}{2} \alpha_\mu \left[1 + \frac{\alpha_\mu}{3\pi} \frac{2}{\varepsilon} \right]$$

$$\Rightarrow \mu^2 \frac{d\alpha_\mu}{d\mu^2} = -\frac{\varepsilon}{2} \alpha_\mu \left[1 - \frac{\alpha_\mu}{3\pi} \frac{2}{\varepsilon} + o(\alpha_\mu^2) \right]$$

$$\Rightarrow \mu^2 \frac{d\alpha}{d\mu^2} = -\frac{\varepsilon}{2} \alpha_\mu + \frac{\alpha_\mu^2}{3\pi} \Rightarrow \text{take } \varepsilon \to 0 \text{ limit} \Rightarrow$$

$$\Rightarrow \boxed{\mu^2 \frac{d\alpha_\mu}{d\mu^2} = \frac{\alpha_\mu^2}{3\pi}}$$ ~ renormalization group (RG) equation

Def. Beta-function of a theory: $\boxed{\beta(\alpha) \equiv \mu^2 \frac{d\alpha}{d\mu^2}}$

In QED the beta-function is $\boxed{\beta_{QED}(\alpha) = \frac{\alpha^2}{3\pi}}$.

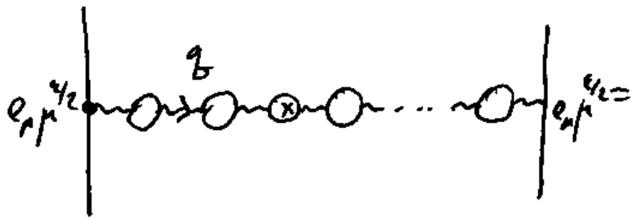
Solve $\frac{d\alpha}{d\ln\mu^2} = \frac{\alpha^2}{3\pi} \Rightarrow \frac{d\alpha}{\alpha^2} = \frac{1}{3\pi} d\ln\mu^2 \Rightarrow$

$$\Rightarrow -\frac{1}{\alpha}\Big|_{\alpha_\mu}^{\alpha(Q^2)} = \frac{1}{3\pi} \ln\mu^2 \Big|_{\mu^2}^{Q^2} \Rightarrow -\frac{1}{\alpha(Q^2)} + \frac{1}{\alpha_\mu} = \frac{1}{3\pi} \ln\left(\frac{Q^2}{\mu^2}\right)$$

$$\Rightarrow \boxed{\alpha(Q^2) = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi} \ln\left(\frac{Q^2}{\mu^2}\right)}}$$ ~ running of QED coupling (like $\alpha_{eff}(Q^2)$ before).

Alternative derivation of QED beta-function:

dressed Coulomb potential



$$\Rightarrow \tilde{V}(\vec{q}) \propto \frac{\alpha_\mu \mu^\varepsilon}{1 - \Pi_2^{ren}(-\vec{q}^{\,2})} \, .$$

Dropping the mass for simplicity and using $\Pi_2^{\overline{MS}}$ from p. 215 we get

$$\tilde{V}(\vec{q}) \propto \frac{\alpha_\mu \mu^\varepsilon \simeq 1}{1 - \frac{\alpha_\mu}{3\pi}\left[\ln\frac{\vec{q}^{\,2}}{\mu^2} - \frac{5}{3}\right]} \, .$$

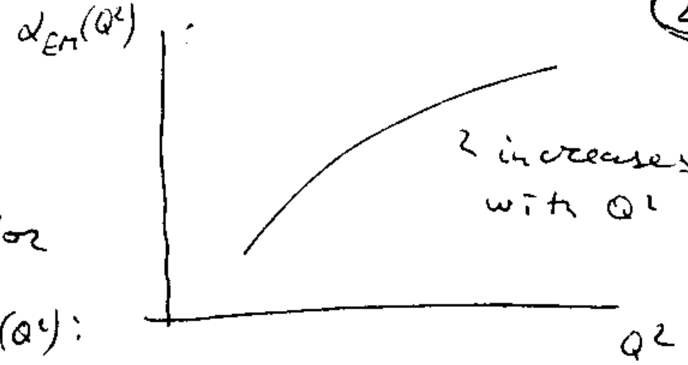
$\tilde{V}(\vec{q})$ is an observable $\Rightarrow$ $0 = \mu^2 \frac{d}{d\mu^2} \tilde{V}(\vec{q})$ $\Rightarrow$

$$\Rightarrow \frac{\beta(\alpha)}{1 - \frac{\alpha_\mu}{3\pi}\left[\ln\frac{\vec{q}^{\,2}}{\mu^2} - \frac{5}{3}\right]} - \frac{\alpha_\mu\left(-\frac{1}{3\pi}\beta(\alpha)\left(\ln\frac{\vec{q}^{\,2}}{\mu^2} - \frac{5}{3}\right) + \frac{\alpha_\mu}{3\pi}\right)}{\left[1 - \frac{\alpha_\mu}{3\pi}\left(\ln\frac{\vec{q}^{\,2}}{\mu^2} - \frac{5}{3}\right)\right]^2} = 0$$

$$\Rightarrow \boxed{\beta_{QED}(\alpha) = \frac{\alpha_\mu^2}{3\pi}}$$

$\Rightarrow$ get the same β-function for QED using a different procedure.

We can plot the coupling:



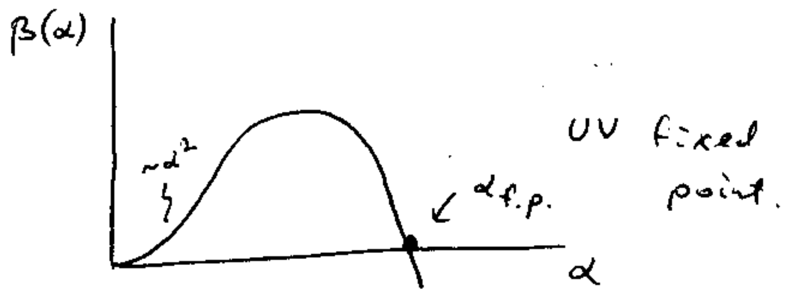
Note a problem: denominator may become 0, giving ∞ $\alpha(Q^2)$:

$$1 = \frac{\alpha_r}{3\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right) \Rightarrow \Lambda^2 = \mu^2 e^{3\pi/\alpha_r}$$

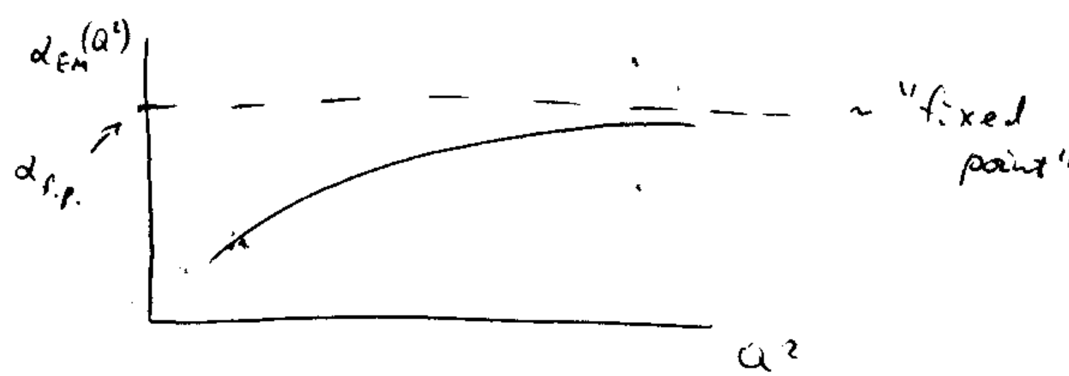
$$\Rightarrow \boxed{Q^2 = \mu^2 e^{3\pi/\alpha}}$$ ~ Landau singularity

(QED is incomplete, gets modified in the UV)

The full QED beta-function may look like



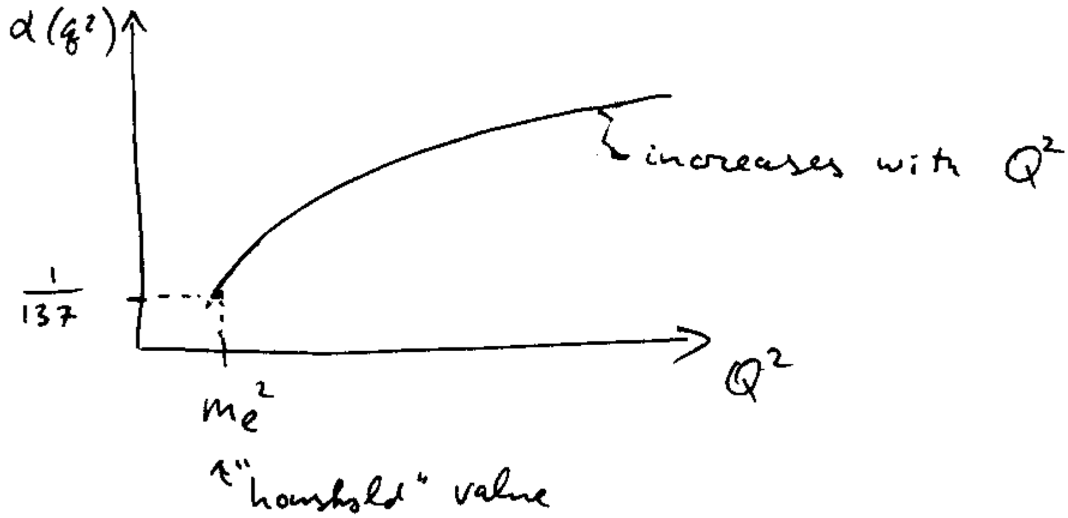
Then



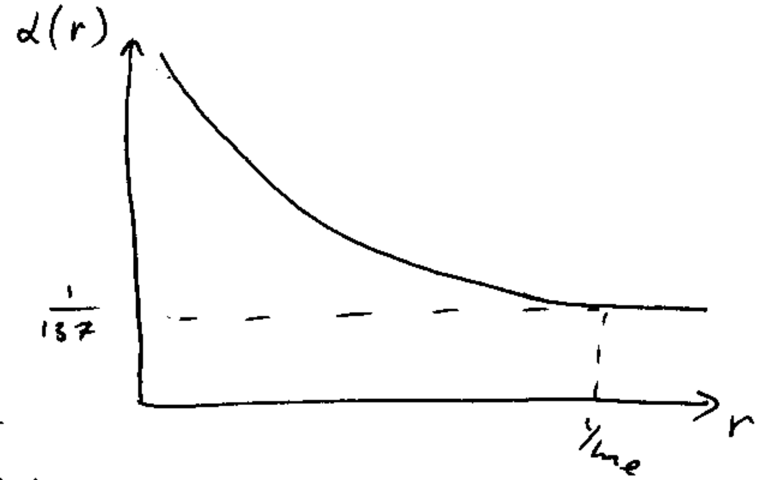
$$\Lambda_{QED} = m_e e^{\frac{1}{2}\frac{3\pi}{\alpha_{EM}}} \simeq 10^{280}\ \text{MeV} = 10^{277}\ \text{GeV},\ \text{vs}$$

$$M_{\text{Planck}} \simeq 10^{19}\ \text{GeV}$$

$$\Rightarrow \quad \alpha(Q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln\left(\frac{Q^2}{m^2 e^{5/3}}\right)}$$



In coordinate space:



Electron-positron pairs pop out of the vacuum to screen the effective charge of the electron, just like in a dielectric: