Brief Review of 1st Semester

We considered various classical field theories for particles with spin $\emptyset, \frac{1}{2}, 1, \ldots$

Spin-$\emptyset$: $$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2 - U(\varphi)$$

$U(\varphi) = 0$ ~ free scalar field theory

$$\left.\begin{array}{l} U(\varphi) = \frac{\lambda}{3!}\varphi^3 \sim \varphi^3\text{-theory} \\ U(\varphi) = \frac{\lambda}{4!}\varphi^4 \sim \varphi^4\text{-theory} \end{array}\right\} \text{interacting theories.}$$

~ talked about symmetries & conservation laws for classical field theories

<u>Spin-1/2:</u> $\boxed{\mathcal{L} = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi}$

free Dirac Lagrangian

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \sim \text{Dirac spinor}$$

$\gamma^\mu \sim$ Dirac matrices, $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$.

(Dirac representation)

<u>Spin-1:</u> $\boxed{\mathcal{L}_{gauge} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$ free vector field

$A_\mu \sim$ vector field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Interactions: QED

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

$D_\mu = \partial_\mu + ie A_\mu$ ~ covariant derivative

We <u>quantized</u> free fields: scalar, spinor, vector:

e.g. scalar field: promote φ, $\pi = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}}$ to operators with

$$\boxed{\begin{gathered}[\varphi(\vec{x},t), \pi(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y}) \\ [\varphi(\vec{x},t), \varphi(\vec{y},t)] = [\pi(\vec{x},t), \pi(\vec{y},t)] = 0\end{gathered}}$$

Postulating that the Hamiltonian H generates time evolution got $[\Box + m^2]\varphi = 0 \Rightarrow$ K-G equation for operator φ $\Rightarrow$ solved if

$$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^+_{\vec{k}}\, e^{ik\cdot x} \right]$$

$$\Rightarrow \left[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'} \right] = (2\pi)^3\, 2\varepsilon_k\, \delta(\vec{k} - \vec{k}')$$

$$\left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'} \right] = \left[\hat{a}^+_{\vec{k}}, \hat{a}^+_{\vec{k}'} \right] = 0.$$

$$H = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\, \varepsilon_k\, \hat{a}^+_{\vec{k}}\, \hat{a}_{\vec{k}}$$

$|0\rangle$ ~ vacuum

$$\left.\begin{array}{l} \hat{a}^+_{\vec{k}}|0\rangle \text{ ~ one-particle state} \\ \hat{a}^+_{\vec{k}_1}\hat{a}^+_{\vec{k}_2}|0\rangle \text{ ~ 2-particle state} \\ \quad\vdots \qquad\qquad \vdots \end{array}\right\} \text{Fock states}$$

Similar canonical quantization can be carried out for vector fields (A_μ) & spinor fields (ψ), except need anti-commutators for ψ.

Correlators in Free Field Theories:

$$D(x-y) \equiv \langle 0| \varphi(x)\varphi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} 2\pi\, \delta^{(+)}(k^2-m^2),$$

Time-ordered product:

$$T\varphi(x)\varphi(y) \equiv \theta(x^0-y^0)\varphi(x)\varphi(y) + \theta(y^0-x^0)\varphi(y)\varphi(x)$$

Feynman propagator:

$$\boxed{D_F(x-y) \equiv \langle 0|T\varphi(x)\varphi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2-m^2+i\varepsilon}.}$$

$$(\Box + m^2) D_F(x-y) = -i\,\delta^4(x-y)$$

Dirac field:

$$\boxed{S_F(x-y) = \langle 0|T\psi(x)\bar\psi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i(\gamma\cdot k+m)}{k^2-m^2+i\varepsilon}}$$

Vector field: $(m=0)$

$$\boxed{D_{\mu\nu}(x-y) = \langle 0|T A_\mu(x)A_\nu(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-i}{k^2+i\varepsilon} \cdot \left[g_{\mu\nu} + (\lambda-1)\frac{k_\mu k_\nu}{k^2}\right].}$$

$\lambda=1$ Feynman gauge

$\lambda=0$ Landau gauge

Interacting Fields and Feynman Diagrams (cont'd)

Interaction Picture & Correlation Functions (cont'd)

consider φ^4 - theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4.$$

Work in the interaction picture: $H = H_0 + H_{int}$ (H, S)

$$\Rightarrow \quad \hat{O}_I(\vec{x},t) = e^{iH_0(t-t_0)}\hat{O}_S(\vec{x})\,e^{-iH_0(t-t_0)}$$

$$|\psi(t)\rangle_I = U(t,t')\,|\psi(t')\rangle_I$$

$\Rightarrow (\Box + m^2)\varphi_I = 0$ like free field th'y

where $U(t,t_0) = e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}$

Ham. in H. or S. picture,

$$i\partial_t U(t,t') = H_I(t)\,U(t,t')$$

with $H_I(t) = e^{iH_0(t-t_0)}H_{int}\,e^{-iH_0(t-t_0)}$.

$$U(t,t') = T\exp\left\{-i\int_{t'}^{t} dt''\,H_I(t'')\right\}$$

~ unitary time-evolution operator.

$U(t_1,t_2)\,U^+(t_1,t_2) = \mathbb{1}$

$U^+(t_1,t_2) = U(t_2,t_1)$

$$H_I(t) = \int d^3x\,\frac{\lambda}{4!}\varphi_I^4$$ ~ for φ^4 theory.

What about correlation functions in interacting theory?

We showed that

$$\langle \Psi_0 | T\, \varphi_H(x)\, \varphi_H(y) | \Psi_0 \rangle = \frac{\langle 0 | T\{ \varphi_I(x)\, \varphi_I(y)\, e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} \} | 0 \rangle}{\langle 0 | T e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} | 0 \rangle}$$

and that this is true in general:

$$\langle \Psi_0 | T\{ \varphi_H(x_1) \dots \varphi_H(x_n) \} | \Psi_0 \rangle = \frac{\langle 0 | T\{ \varphi_I(x_1) \dots \varphi_I(x_n)\, e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} \} | 0 \rangle}{\langle 0 | T e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} | 0 \rangle}$$

Gell-Mann-Low f-la

$|\Psi_0\rangle$ ~ vacuum of interacting th'y

$|0\rangle$ ~ vacuum of free th'y (perturbative vacuum).

(Note that $|\Psi_0\rangle \neq |0\rangle$ in general.)

Wick's Theorem

(Def.) Normal ordering ~ move all $\hat{a}^+$ left of all $\hat{a}$.

$$:\hat{a}_{\vec{k}}\,\hat{a}^+_{\vec{p}}: \;=\; \hat{a}^+_{\vec{p}}\,\hat{a}_{\vec{k}}$$

(Def.) Contraction $\overbrace{\varphi(x)\,\varphi(y)} = T\varphi(x)\varphi(y) - :\varphi(x)\varphi(y):$

Note that $\overbrace{\varphi(x)\,\varphi(y)} = D_F(x-y) = \langle 0|T\varphi(x)\varphi(y)|0\rangle$

~ contraction is propagator

Wick's th'm main consequence:

$$\langle 0|T\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)|0\rangle = \overbrace{\varphi_1\,\varphi_2}\;\overbrace{\varphi_3\,\varphi_4}\;\ldots\;\overbrace{\varphi_{n-1}\,\varphi_n}$$

+ other perm's (even n only)

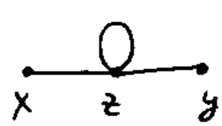
Feynman Rules for φ^4-theory

Using Gell-Mann–Low f-la and Wick's theorem we can evaluate correlators order-by-order in λ:

$$\langle\Psi_0|T\varphi_H(x)\varphi_H(y)|\Psi_0\rangle = \frac{\langle 0|T\varphi_I(x)\varphi_I(y)\,e^{-i\frac{\lambda}{4!}\int d^4z\,\varphi_I^4(z)}|0\rangle}{\langle 0|T\,e^{-i\frac{\lambda}{4!}\int d^4z'\,\varphi_I^4(z')}|0\rangle}$$

Numerator $= \langle 0|T\varphi(x)\varphi(y)|0\rangle - i\frac{\lambda}{4!}\int d^4z\,\langle 0|T\varphi(x)\varphi(y)$

$\varphi^4(z)|0\rangle + \ldots =$ 

$- i\lambda\Big[\frac{1}{2}$ 

$+ \frac{1}{8}$ 

$\Big]$

$+ O(\lambda^2)$

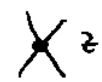
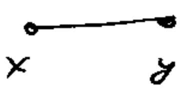
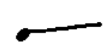
connected　　vacuum bubble

$\Rightarrow$ ditto for denominator

One can show that

$$\boxed{\langle \Psi_0 | T\, \varphi_H(x_1)\, \varphi_H(x_2) \ldots \varphi_H(x_n) | \Psi_0 \rangle = \langle 0 | T\, \varphi(x_1)\, \varphi(x_2) \ldots \varphi(x_n) \cdot e^{-i \frac{\lambda}{4!} \int d^4 z\, \varphi^4(z)} | 0 \rangle_{\text{connected}}}$$

Feynman rules for φ^4 theory (Green fcts, coord. space)

1. Draw all connected diagrams.
2. Each vertex gives $-i\lambda \int d^4 z =$ 
3. Each propagator gives $D_F(x-y) =$ 
4. Include symmetry factors.
5. Each external vertex gives  $= 1$.

Fourier-transform to momentum space:

$$\tilde{G}(p_1, p_2, \ldots, p_n) = \int d^4x_1\, d^4x_2 \ldots d^4x_n\, e^{ip_1 \cdot x_1 + \ldots + ip_n \cdot x_n} \cdot G(x_1, x_2, \ldots, x_n)$$

$\Rightarrow$ momentum-space Feynman rules

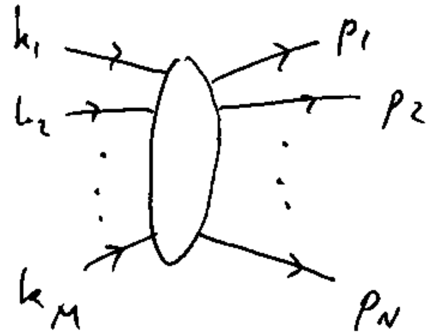
Cross Sections, $\mathcal{S}$-Matrix & the Reduction Formula (A9)

$\mathcal{S}$ ~ time evolution operator, $|\psi_f\rangle = \mathcal{S}|\psi_i\rangle$

$$\boxed{\mathcal{S} = \mathbb{1} + iT}$$

$$\langle \{p_i\} | iT | \{k_j\} \rangle = i\mathcal{M}(\{k_j\} \to \{p_i\}) (2\pi)^4 \delta^4\Big(\sum_i p_i - \sum_j k_j\Big)$$

scattering amplitude



Cross section $2 \to n$:

$$d\sigma = \frac{1}{2E_{k_1}\, 2E_{k_2}\, |\vec{v}_1 - \vec{v}_2|} \frac{1}{n!} \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_{p_i}} |\mathcal{M}_{2\to n}|^2 (2\pi)^4 \delta^4\Big(k_1 + k_2 - \sum_{j=1}^{n} p_j\Big)$$

Decay rate ~ similar

LSZ reduction f-la for a scalar theory: to find $\mathcal{M}_{2\to n}$ from G_{n+2} (Green function), remove propagators of external legs & put external momenta on mass-shell.

Feynman Rules for Scattering Amplitudes in φ^4-Theory

(A10)

① Each internal line $\xrightarrow{k} \; = \dfrac{i}{k^2-m^2+i\varepsilon}$

② Each vertex $\times = -i\lambda$

③ External lines give 1.

④ Impose 4-momentum conservation at each vertex. Integrate over each indep. internal momentum $\dfrac{d^4k}{(2\pi)^4}$.

⑤ Divide by symmetry factors.

⑥ Connected diagrams only.

QED: Tree-Level Processes

$$\mathcal{L}_{QED} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad D_\mu = \partial_\mu + ieA_\mu$$

Feynman Rules for fermions

① $\alpha \xrightarrow{k} \beta \qquad \dfrac{i(\not{k}+m)_{\beta\alpha}}{k^2-m^2+i\varepsilon}$

② $\bullet\!\xrightarrow{p}\sigma \quad \bar{u}_\sigma(p) \qquad\qquad \bullet\!\xrightarrow{p}\sigma \;(\leftarrow) \quad v_\sigma(p)$

$\sigma\xrightarrow{p}\!\bullet \quad u_\sigma(p) \qquad\qquad \sigma\xrightarrow{p}\!\bullet \;(\leftarrow) \quad \bar{v}_\sigma(p)$

③ Signs! (-1) for loops, line that begins & ends in initial state, ...

$$\sum_\sigma u_\sigma(k)\,\bar{u}_\sigma(k) = \not{k}+m \;, \quad \sum_\sigma v_\sigma(k)\,\bar{v}_\sigma(k) = \not{k}-m$$