Analytic Continuation of $\zeta(z)$:

Start with the Γ - function:

$$\Gamma(z) = \int_0^\infty dt \cdot t^{z-1} e^{-t} = \Big|^{t \to nt} = \int_0^\infty dt \cdot t^{z-1} \cdot n^z e^{-nt}$$

Remember that $\boxed{\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z}}$ => we obtain that

$$\Rightarrow \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{\Gamma(z)} \cdot \int_0^\infty dt \cdot t^{z-1} e^{-nt} = \frac{1}{\Gamma(z)} \int_0^\infty dt \, \frac{t^{z-1}}{e^t - 1}$$

$$\Rightarrow \quad \boxed{\Gamma(z)\, \zeta(z) = \int_0^\infty dt \, \frac{t^{z-1}}{e^t - 1}}$$

Can use this integral to define $\zeta(z)$ beyont its "naive" domain of $Re\, z > 1$. For instance, to extend the definition of $\zeta(z)$ to $Re\, z > -2$ split the integral:

$$\zeta(z)\,\Gamma(z) = \int_0^1 dt \, \frac{t^{z-1}}{e^t - 1} + \underbrace{\int_1^\infty dt \, \frac{t^{z-1}}{e^t - 1}}_{\text{defined for } \forall z}$$

In the first term write:

$$\frac{1}{e^t - 1} = \underbrace{\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right)}_{o(t^2)} + \frac{1}{t} - \frac{1}{2} + \frac{t}{12}$$

such that

$$\zeta(z)\Gamma(z) = \underbrace{\int_0^1 dt\, t^{z-1} \underbrace{\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{2}\right)}_{O(t^2)}}_{O(t^{z+1}) \Rightarrow \text{ finite for } \mathrm{Re}\, z > -2.} + \frac{1}{z-1} - \frac{1}{2z} + \frac{1}{12(z+1)}$$

$$+ \int_1^\infty dt\, \frac{t^{z-1}}{e^t-1} \quad \Rightarrow \quad \zeta(z) \text{ defined this way is finite for } \mathrm{Re}\, z > -2.$$

$\Rightarrow$ can do this for $\forall$ integer n: expand $\frac{1}{e^t-1}$ up to $O(t^n)$

$\Rightarrow$ define $\zeta(z)$ for $\mathrm{Re}\, z > -n$.

Near $z \simeq 0$: $\quad \zeta(0)\underbrace{\Gamma(0)}_{\Gamma(z) \simeq \frac{1}{z}} \simeq -\frac{1}{2\cdot z} + \text{finite}$

$$\Rightarrow \zeta(0) = -1/2 \;;$$

near $z \simeq -1$: $\quad \zeta(-1)\underbrace{\Gamma(-1+\varepsilon)}_{\simeq -\frac{1}{\varepsilon}} = \frac{1}{12\varepsilon} + \text{finite}$

$$\Rightarrow \zeta(-1) = -\frac{1}{12}.$$