(A1)

Def. $\vec{N}_{\pm}^{old} \equiv \frac{1}{2}\left(\vec{L} \pm i\vec{K}\right)$, with $\vec{L} = -i\vec{x}\times\vec{\nabla}$,

$$\vec{K} = -i\left[x^0\vec{\nabla} + \vec{x}\,\partial_0\right].$$

$N_{\pm}^{old}$ satisfy the following commutation relations:

$$\left.\begin{cases} [N_+^i, N_+^j] = i\varepsilon^{ijk} N_+^k & \Rightarrow SU(2) \\ [N_-^i, N_-^j] = i\varepsilon^{ijk} N_-^k & \Rightarrow SU(2) \\ [N_+^i, N_-^j] = 0 & \end{cases}\right\} \Rightarrow \boxed{SO(3,1) \Leftrightarrow SU(2)\otimes SU(2)}$$

In the tensor notation, to introduce spin define

$$\boxed{J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}}$$

with $[L_{\mu\nu}, S_{\rho\sigma}] = 0$ and $S_{\mu\nu}$ satisfying the same commutation relations as $L_{\mu\nu}$.

In the 3-vector notation this corresponds to defining

$$\vec{N}_+ \equiv \vec{N}_+^{old} + \vec{S}_+ \,, \quad \vec{N}_- \equiv \vec{N}_-^{old} + \vec{S}_-$$

where $[S_{\pm}^i, S_{\pm}^j] = i\varepsilon^{ijk} S_{\pm}^k$, $[S_+^i, S_-^j] = 0$

(the latter is due to S_+ & S_- acting on different spin "spaces"). The new $\vec{N}_+, \vec{N}_-$ satisfy the same above commutation relations as $\vec{N}_+^{old}, \vec{N}_-^{old}$.

A2

What happens to $\mathcal{U}(\Lambda) = e^{i\vec{\xi}\cdot\vec{K} - i\vec{\theta}\cdot\vec{L}}$ with introduction of spin? To see it we first rewrite $\mathcal{U}(\Lambda)$ in terms of $\vec{N}_{\pm}^{old}$: as $\vec{L} = \vec{N}_+^{old} + \vec{N}_-^{old}$, $i\vec{K} = \vec{N}_+^{old} - \vec{N}_-^{old}$

$$\mathcal{U}(\Lambda) = e^{\vec{N}_+^{old}\cdot(\vec{\xi} - i\vec{\theta}) - \vec{N}_-^{old}\cdot(\vec{\xi} + i\vec{\theta})}$$

after the shift of $\vec{N}_{\pm}^{old}$ by spin operators we get

$$\boxed{\mathcal{U}(\Lambda) = e^{\vec{N}_+\cdot(\vec{\xi} - i\vec{\theta}) - \vec{N}_-\cdot(\vec{\xi} + i\vec{\theta})}}$$

This means that the field in a particular representation of Lorentz group transforms as

$$F(x) \to F'(x') = \mathcal{U}(\Lambda)\, F(x') = e^{\vec{S}_+\cdot(\vec{\xi} - i\vec{\theta}) - \vec{S}_-\cdot(\vec{\xi} - i\vec{\theta})} F(x). \qquad (*)$$

any field

<u>Example</u> For $\vec{S}_+ = 0$, $\vec{S}_- = 0$, as is the case for the scalar field $\varphi(x)$ corresponding to $(0,0)$ representation, we get $\varphi(x) \to \varphi'(x') = \varphi(x)$ as expected!

Note that the net spin operator is $\vec{S} = \vec{S}_+ + \vec{S}_-$.

Lorentz group representations can be classified by the values of s_+, s_- with the eigenvalues of $\vec{S}_+^2, \vec{S}_-^2$ being $s_+(s_+ + 1)$ and $s_-(s_- + 1)$.

" $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

For left-handed spinors $(\frac{1}{2},0)$ we have 2 dof $\Rightarrow$

$\Rightarrow$ $\chi_L = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ ~ a 2-component object.

To find out how it transforms under Lorentz transformations we put $\vec{S}_+ = \frac{1}{2}\vec{\sigma}$, $\vec{S}_- = 0$ (due to $(\frac{1}{2},0)$) into Eq. $(*)$ on the previous page.

We get

$$\chi_L(x) \to \chi_L'(x') = \underbrace{e^{\frac{\vec{\sigma}}{2}\cdot(\vec{\xi}-i\vec{\theta})}}_{\equiv \Lambda_L} \chi_L(x)$$

$$\Rightarrow \boxed{\Lambda_L = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})}}$$

Similarly, the right-handed $(0,\frac{1}{2})$ spinors transform as $(\vec{S}_+ = 0,\ \vec{S}_- = \frac{1}{2}\vec{\sigma}$ in Eq. $(*)$)

$$\chi_R(x) \to \chi_R'(x') = \underbrace{e^{-\frac{\vec{\sigma}}{2}\cdot(\vec{\xi}-i\vec{\theta})}}_{\equiv \Lambda_R} \chi_R(x)$$

$$\Rightarrow \boxed{\Lambda_R = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})}}$$

For Dirac spinors $\psi(x) = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$ one can show that they transform as

$$\psi(x) \to \psi'(x') = e^{-\frac{i}{2}\omega^{\mu\nu} S_{\mu\nu}} \psi(x)$$ (standard f-la, true for all fields)

with $\boxed{S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]}$ and γ^μ the Dirac matrices.

Defining $K^i = L^{0i} + S^{0i} = L^{0i} - \frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

and $J^i = \frac{1}{2}\varepsilon^{ijk}(L^{jk} + S^{jk}) = L^i + \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$

we see that $\left(\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)\right)$

$$\vec{N}_+ = \frac{1}{2}(\vec{J} + i\vec{K}) = \vec{N}_+^{old} + \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & 0 \end{pmatrix} \equiv \vec{N}_+^{old} + \vec{S}_+$$

$$\vec{N}_- = \frac{1}{2}(\vec{J} - i\vec{K}) = \vec{N}_-^{old} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \equiv \vec{N}_-^{old} + \vec{S}_-$$

such that $\boxed{\vec{S}_+ = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{S}_- = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \vec{\sigma} \end{pmatrix}}$

are really two operators acting onto different spaces, the left- and right-handed spinors' spaces.