Take ${}^{j}\langle Q_f, T_f | Q_i, T_i \rangle^{j} = \int_{-\infty}^{\infty} dq\, dq' \langle Q_f, T_f | q', t' \rangle \cdot$

$\cdot {}^{j}\langle q', t' | q, t \rangle^{j} \langle q, t | Q_i, T_i \rangle \qquad , \; T_f > t' > t > T_i.$

Now, ${}_H\langle Q, T | q, t \rangle_H = {}_S\langle Q(T) | e^{-\frac{i}{\hbar}\hat{H}T} e^{\frac{i}{\hbar}\hat{H}t} | q(t) \rangle_S =$

$= \sum_m {}_S\langle Q(T) | e^{-\frac{i}{\hbar}\hat{H}T} | m \rangle \langle m | e^{\frac{i}{\hbar}\hat{H}t} | q(t) \rangle_S =$

(here $\hat{H}|m\rangle = E_m |m\rangle$ ~ Hamiltonian eigenstates)

${}_S\langle q(t) | m \rangle \equiv \phi_m(q,t)$ ~ corresponding wave function

$\Rightarrow \; {}_H\langle Q, T | q, t \rangle_H = \sum_m \phi_m(Q,T) \phi_m^*(q,t) e^{-\frac{i}{\hbar}E_m(T-t)}$

$\Rightarrow \; {}^{j}\langle Q_f, T_f | Q_i, T_i \rangle^{j} = \int_{-\infty}^{\infty} dq\, dq' \sum_{m,n} \phi_m(Q_f)\, \phi_m^*(q') \cdot \phi_n(q) \cdot$

$\cdot \phi_n^*(Q_i)\, e^{-\frac{i}{\hbar}E_m(T_f - t')} \cdot e^{\frac{i}{\hbar}E_n(T_i - t)}\; {}^{j}\langle q', t' | q, t \rangle^{j}$

Want to take $T_i \to -\infty$, $T_f \to +\infty$, but we want to have ground states at $\pm\infty \Rightarrow$ take $T_i \to -\infty e^{-i\delta} = -\infty(1-i\delta)$

$T_f \to +\infty e^{-i\delta} = +\infty(1-i\delta)$, δ ~ infinitesimal

$\Rightarrow$ pick out vacuum states $m=0$ and $n=0$ at $t=\pm\infty$

$$\lim_{\substack{T_i \to -\infty(1-i\delta) \\ T_f \to +\infty(1-i\delta)}} {}^{j}\langle Q_f, T_f | Q_i, T_i \rangle^{j} = \int_{-\infty}^{\infty} dq\, dq'\, \phi_0(Q_f)\, \phi_0^*(q')\, \phi_0(q) \cdot \phi_0^*(Q_i)\, e^{-\frac{i}{\hbar}E_0(T_f - T_i)}\; {}^{j}\langle q', t' | q, t \rangle^{j}$$

$\Rightarrow$ vacuum-to-vacuum transition amplitude

is $\langle 0, t_f | 0, t_i \rangle^j = \int dq_i\, dq_f \; \phi_0^*(q_f, t_f) \langle q_f, t_f | q_i, t_i \rangle^j$

$$\cdot\, \phi_0(q_i, t_i) = \lim_{\substack{T_f \to +\infty e^{-i\delta} \\ T_i \to -\infty e^{-i\delta}}} \frac{\langle Q_f, T_f | Q_i, T_i \rangle^j}{\phi_0(Q_f)\, \phi_0^*(Q_i)\, e^{-\frac{i}{\hbar} E_0 (T_f - T_i)}}$$

$$\Rightarrow \langle 0, +\infty | 0, -\infty \rangle^j \propto \lim_{\substack{T_f \to +\infty(1-i\delta) \\ T_i \to -\infty(1-i\delta)}} \langle Q_f, T_f | Q_i, T_i \rangle$$

$\Rightarrow$ get the phases for time variable

Note that $|0, t_i\rangle_H = \int dq_i \; |q_i, t_i\rangle_H \underbrace{{}_H\langle q_i, t_i | 0 \rangle_H}_{\phi_0(q_i, t_i)}$

$$\Rightarrow |0, t_i\rangle_H = \int dq_i \; |q_i, t_i\rangle_H \; \phi_0(q_i, t_i)$$

$\hat{q}_H(t_i)|0, t_i\rangle_H = \int dq_i \cdot q_i \cdot |q_i, t_i\rangle_H \; \phi_0(q_i, t_i) \neq 0$, not an eigenstate of $\hat{q}_H(t_i)$!

alternatively:
$\Rightarrow$ Vacuum-to-vacuum transition amplitude:

$$\langle \Psi_0^{out} | \Psi_0^{in} \rangle = \int dq_i \, dq_f \, \langle \Psi_0^{out} | q_f, t_f \rangle_H \; {}_H\langle q_f, t_f | q_i, t_i \rangle_H \cdot$$

$$\cdot {}_H\langle q_i, t_i | \Psi_0^{in} \rangle = \int dq_i \, dq_f \, \phi_0^*(q_f, t_f) \, \phi_0(q_i, t_i) \, {}_H\langle q_f, t_f | q_i, t_i \rangle_H .$$

In preparation for field theory, consider harmonic oscillator: its ground state is known,

$$\phi_0(q) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\frac{m\omega}{2\hbar} q^2}$$

$$\Rightarrow \langle \Psi_0^{out} | \Psi_0^{in} \rangle = \int dq_i \, dq_f \, \sqrt{\frac{m\omega}{\hbar\pi}} \, e^{-\frac{m\omega}{2\hbar}\left(q_f^2 + q_i^2\right)} .$$

$$\cdot {}_H\langle q_f, t_f | q_i, t_i \rangle_H = \left| \begin{array}{l} \text{as for } \forall \text{ smooth } f(t): \\ f(t_f) + f(t_i) = \lim\limits_{\varepsilon \to 0+} \varepsilon \int_{t_i}^{t_f} dt \, f(t) \, e^{-\varepsilon |t|} \end{array} \right.$$

$$= \lim_{\varepsilon \to 0} \int dq_i \, dq_f \, \sqrt{\frac{m\omega}{\hbar\pi}} \, e^{-\frac{m\omega}{2\hbar} \varepsilon \int_{\omega t_i}^{\omega t_f} d(\omega t) \, q^2(t) \, e^{-\varepsilon\omega |t|}} \cdot {}_H\langle q_f, t_f | q_i, t_i \rangle_H$$

$$\simeq \sqrt{\frac{m\omega}{\hbar\pi}} \int dq_i \, dq_f \, e^{-\frac{1}{2}\frac{1}{\hbar} m\omega^2 \varepsilon \int_{t_i}^{t_f} dt \, q^2(t)} \underbrace{{}_H\langle q_f, t_f | q_i, t_i \rangle}_{= \int [\mathcal{D}q] \, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \, L}, \; q(t_i) = q_i, \; q(t_f) = q_f}$$

$$= \int [\mathcal{D}q] \, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \cdot \left(L + \frac{1}{2} m\omega^2 i\varepsilon q^2\right)}$$

with $q(t_i)$, $q(t_f)$ not fixed (integrated over).

Taking $t_f \to +\infty$, $t_i \to -\infty$ get

$$\boxed{\langle \psi_0^{out} | \psi_0^{in} \rangle = \int [\mathcal{D}q]\, e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt \left[L + \frac{1}{2} m\omega^2 i\varepsilon q^2 \right]}}$$

where $L = L_{orig} + \hbar j q$, q is not specified at $t = \pm\infty$.

$\Rightarrow$ can add $\frac{i\varepsilon q^2}{2}$ to the Lagrangian instead of rotating time-integration contour.

$\Rightarrow$ similarly add $\frac{i\varepsilon}{2}\varphi^2$ to the $\mathcal{L}$ in field theory.

$\rightsquigarrow$ for more see Weinberg's book, Sect. 9.2 (vol. 1) or Sect. 5.5 in Ryder