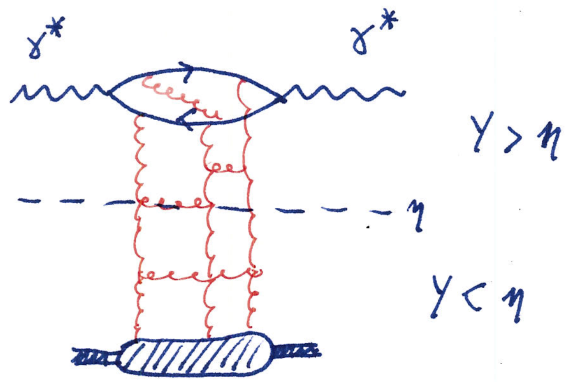
DIS at small-x_B

The hadronic tensor is $\langle B | T\{ J^\mu(x) \; J^\nu(y)\} | B \rangle$

where $J^\mu(x) = \bar{\Psi}(x) \gamma^\mu \Psi(x)$.

The hadronic tensor can be depicted diagrammatically as



The gluon field generated by the target, depicted in red in the figure, is ordered in rapidity. Introduce a cut-off in rapidity that separates slow fields ($Y < \eta$) from fast fields ($Y > \eta$) $\Rightarrow$ Consider the hadronic tensor evaluated in the gluonic external field having rapidity $Y > \eta$

$$\langle T\{ J^\mu(x) \; J^\nu(y)\} \rangle_A$$

The subscript A denotes that the matrix element of the 2 electromagnetic currents is evaluated in the external field of fast gluons with rapidity $Y > \eta$.

NOTATIONS

$$p^{\pm} = \frac{p^0 \pm p^3}{\sqrt{2}}$$ Kogut-Soper convention

Sudakov decomposition

$$p_1^\mu = \frac{\sqrt{s}}{2}(1,0,0,1) \qquad p_2^\mu = \frac{\sqrt{s}}{2}(1,0,0,-1)$$

A generic vector can be written as

$$p^\mu = \alpha\, p_1^\mu + \beta\, p_2^\mu + p_\perp$$

For a given vector A^μ we define

$$A_\bullet \equiv A^\mu p_{1\mu} = \sqrt{\frac{s}{2}}\, A^-$$

$$A_* \equiv A^\mu p_{2\mu} = \sqrt{\frac{s}{2}}\, A^+$$

Rapidity of a particle with momentum p^μ

$$Y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \frac{1}{2} \ln \frac{p^+}{p^-}$$

$$p^\mu = (p^+, p^-, p_\perp) = \frac{\sqrt{2}}{s}\sqrt{m^2 + p_\perp^2}\, e^{Y} p_1^\mu + \frac{\sqrt{2}}{s}\sqrt{m^2 + p_\perp^2}\, e^{-Y} p_2^\mu + p_\perp^\mu$$

Introduce Schwinger's notation

$$\langle x | y \rangle = \delta^{(4)}(x-y)$$

$$\langle x | P_\mu | y \rangle = -i \frac{\partial}{\partial y^\mu} \delta^{(4)}(x-y)$$

$$\langle x | A_\mu | y \rangle = A_\mu(x) \, \delta^{(4)}(x-y)$$

quark propagator : $\langle x | \frac{i}{\not p} | y \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{\not p}$

quark propagator in an external field : $\langle x | \frac{i}{\not P} | y \rangle = \langle x | \frac{i}{\not p + g \not A} | y \rangle$

$|x\rangle$: eigenstates of the coordinate operator $X|x\rangle = x|x\rangle$

$|p\rangle = \int d^4x \, e^{-ip\cdot x} |x\rangle$: eigenstates of the free momentum operator p are plane waves

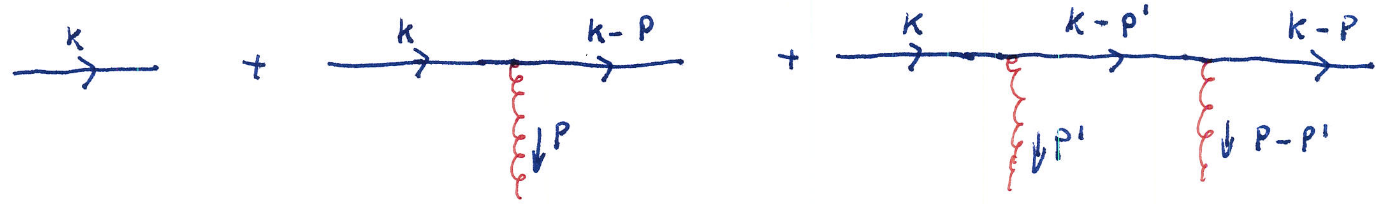
The hadronic tensor can be formally written as

$$\langle T\{J^\mu(x) \, J^\nu(y)\}\rangle_A = \mathrm{Tr}\{ \gamma^\mu \langle x | \frac{i}{\not P} | y \rangle \gamma^\nu \langle y | \frac{i}{\not P} | x \rangle \}$$

$$\langle x| \frac{i}{\not P} |y\rangle = \langle x| \frac{i}{\not P + g\not A} |y\rangle$$

$$\sim \langle x|\frac{i}{\not p}|y\rangle + \langle x|\frac{i}{\not p}\, g\not A(-i)\,\frac{i}{\not p}|y\rangle + \langle x|\frac{i}{\not p}\, g\not A(-i)\,\frac{i}{\not p}\, g\not A(-i)\,\frac{i}{\not p}|y\rangle$$

$\Rightarrow$



Regge limit $\alpha_k \gg \alpha_p$, $\beta_k \ll \beta_p$ $\Rightarrow$ $g^{\mu\nu} = \frac{2}{s}p_1^\mu p_2^\nu + \frac{2}{s}p_1^\nu p_2^\mu + g_\perp^{\mu\nu} \longrightarrow \frac{2}{s}p_1^\mu p_2^\nu$

In momentum space we have

$$\langle k|\frac{1}{\not P}|k-p\rangle = \frac{16\pi^4}{\not k}\delta^{(4)}(p) - \frac{\not k}{k^2+i\epsilon}\, g\,\frac{2}{s}\not p_2 A_\bullet(p)\,\frac{\not k-\not p}{(p-k)^2+i\epsilon} +$$

$$+ g^2\frac{4}{s^2}\int\frac{d^4p'}{16\pi^4}\,\frac{\not k}{k^2+i\epsilon}\,\not p_2 A_\bullet(p')\,\frac{\not k-\not p'}{(p'-k)^2+i\epsilon}\,\not p_2 A_\bullet(p-p')\,\frac{\not k-\not p}{(k-p)^2+i\epsilon} + \dots$$

$$\not p_2\,\frac{\not k-\not p'}{(p-k)^2+i\epsilon}\,\not p_2 = \not p_2\,\frac{(\alpha_k-\alpha'_p)\not p_1 + (\not k-\not p')_\perp}{(\alpha_k-\alpha'_p)(\beta_k-\beta'_p)s-(k-p')_\perp^2+i\epsilon}\,\not p_2$$

$$= \not p_2\,\frac{(\alpha_k-\alpha'_p)s}{(\alpha_k-\alpha'_p)(\beta_k-\beta'_p)s-(k-p')_\perp^2+i\epsilon} \longrightarrow \not p_2\,\frac{1}{-\beta'_p+i\epsilon\alpha_k}$$

⇒ We get the expansion of the path-ordered exponential. for $\alpha_k > 0$

$$\langle k | \frac{1}{\not{P}} | k-p\rangle = \frac{16\pi^4 \delta^{(4)}(p)}{\not k} + \frac{4\pi i}{s}\delta(\alpha_p)\frac{\not k}{k^2+i\epsilon}\not p_2 [U-1](p_\perp)\frac{\not k - \not p}{(k-p)^2+i\epsilon}$$

Where $[U-1](p_\perp) = U(p_\perp) - 4\pi^2\delta^{(2)}(p_\perp)$

$U(p_\perp)$ is the Fourier transf. of $U(x_\perp) = P\{e^{ig\int_{-\infty}^{+\infty}du\, A_\bullet(p_1 u + x_\perp)}\}$

The quark propagator in the external field in coordinate space is

$$\langle\psi(x)\bar\psi(y)\rangle_A\, \theta(x_*)\theta(-y_*) = \int d^4z\,\delta(z_*)\frac{\hat x-\hat z}{2\pi^2(x-z)^4}\hat p_2\, U(z_\perp)\frac{\hat z-\hat y}{2\pi^2(y-z)^4}$$

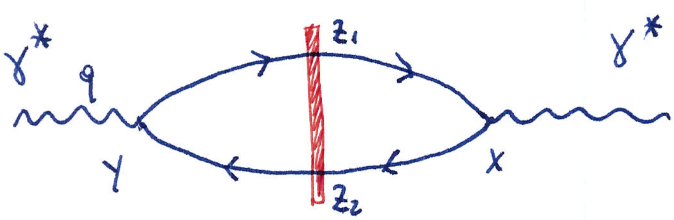
$$\Rightarrow \langle \bar\psi(x)\gamma^\mu\psi(x)\bar\psi(y)\gamma^\nu\psi(y)\rangle_A = \frac{1}{(2\pi^2)^4}\int d^4z_1\, d^4z_2\, \frac{\delta(z_{1*})\delta(z_{2*})}{X_1^4X_2^4Y_1^4Y_2^4}\,\mathrm{tr}\{\gamma^\mu\hat X_1\hat p_2\hat Y_1\gamma^\nu\hat Y_2\hat p_2\hat X_2\}\langle\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}\rangle_A$$

where $U_{z_1}\equiv U(z_{1\perp})$ $\quad \hat x \equiv x_\mu\gamma^\mu \quad X_i = x - z_i$ similar for Y_i

γ^* moment. $q = p_1 - x_B p_2$

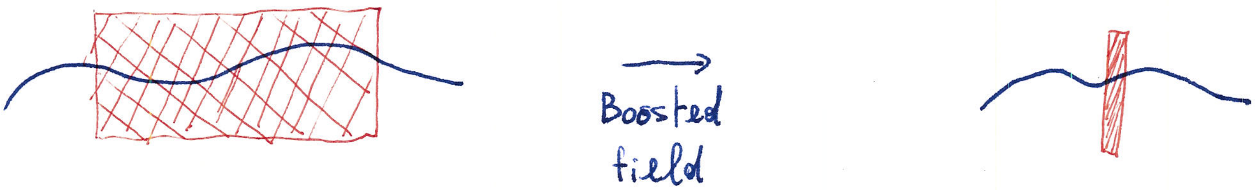
$p = p_2 + \frac{m^2}{s}p_1$

Target momentum



In the spectator frame, the external field is boosted and it is Lorentz contracted and time dilated → shock wave (red strip in the picture)

Spectator frame



Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. After the boost of the field the external field exists only within the infinitely thin wall, so the particles (quarks and gluons) do not have time to deviate in the transverse direction $\Rightarrow$ Replace the gauge factor along the actual path with the one along the straight-line path.

$x_* > 0$, $y_* < 0$

$$\langle \psi(x)\bar{\psi}(y)\rangle_A = \int d^4z\, \delta(z_*) \frac{\hat{x}-\hat{z}}{2\pi^2(x-z)^4} \hat{p}_2\, U(z_\perp) \frac{\hat{z}-\hat{y}}{2\pi^2(y-z)^4}$$

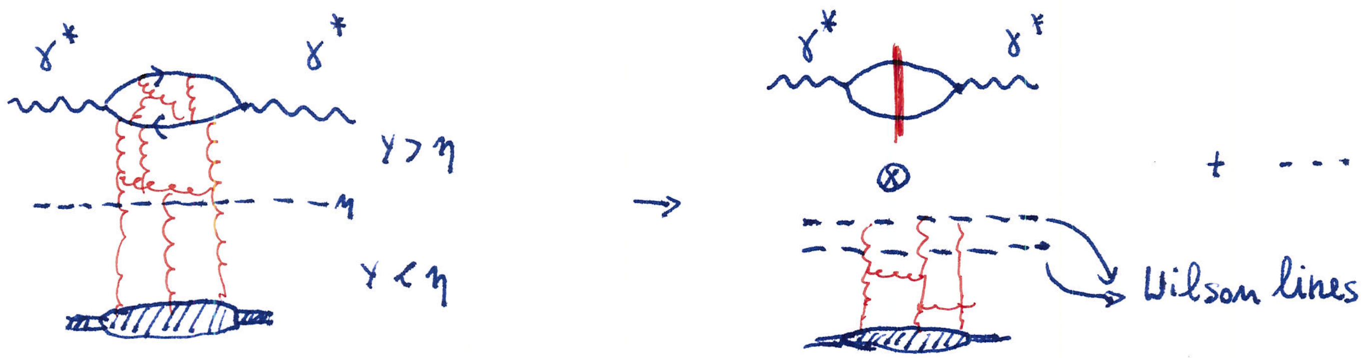
$\hat{x} \equiv x_\mu \gamma^\mu$; $\hat{p}_2 \equiv p_2^\mu \gamma_\mu$; $x_* = x_\mu p_2^\mu$; $x_\bullet = x_\mu p_1^\mu$

$$\langle J^\mu(x) J^\nu(y) \rangle_A = \int d^4z_1\, d^4z_2\, I(z_1, z_2; x, y) \langle \mathrm{tr}\{ U_{z_1} U^\dagger_{z_2} \} \rangle_A + \dots$$

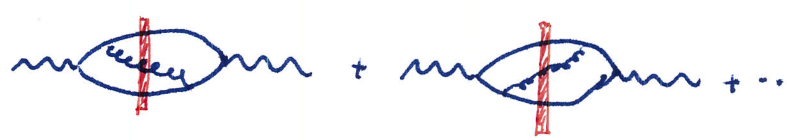
$$\Downarrow$$

$$\langle B | J^\mu(x) J^\nu(y) | B \rangle = \int d^4z_1 d^4z_2\, I(z_1, z_2; x, y) \langle B | \mathrm{tr}\{ U_{z_1} U^\dagger_{z_2} \} | B \rangle + \dots$$

$I(z_1, z_2; x, y)$ is the photon impact factor ; "dots" $\equiv$ higher orders in α_μ



Sample of higher order diagrams for the NLO Impact factor

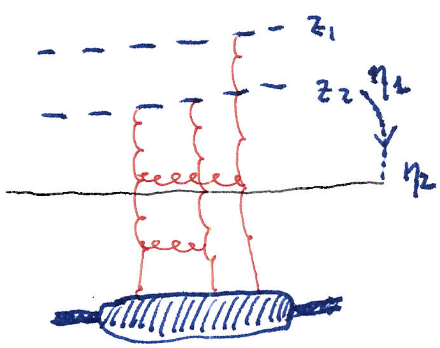


Evolution of Wilson lines

Background field method

$$\langle B|\, \mathrm{tr}\{U_{z_1} U^\dagger_{z_2}\}\, |B\rangle \longrightarrow \langle \mathrm{tr}\{U_{z_1} U^\dagger_{z_2}\}\rangle_A$$

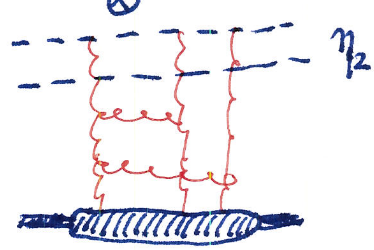
Introduce a cut-off in rapidity which separates quantum fluctuations from classical field; integrate over the quantum field



$\longrightarrow$

$\alpha_s(\eta_1 - \eta_2)\, K$

$\otimes$

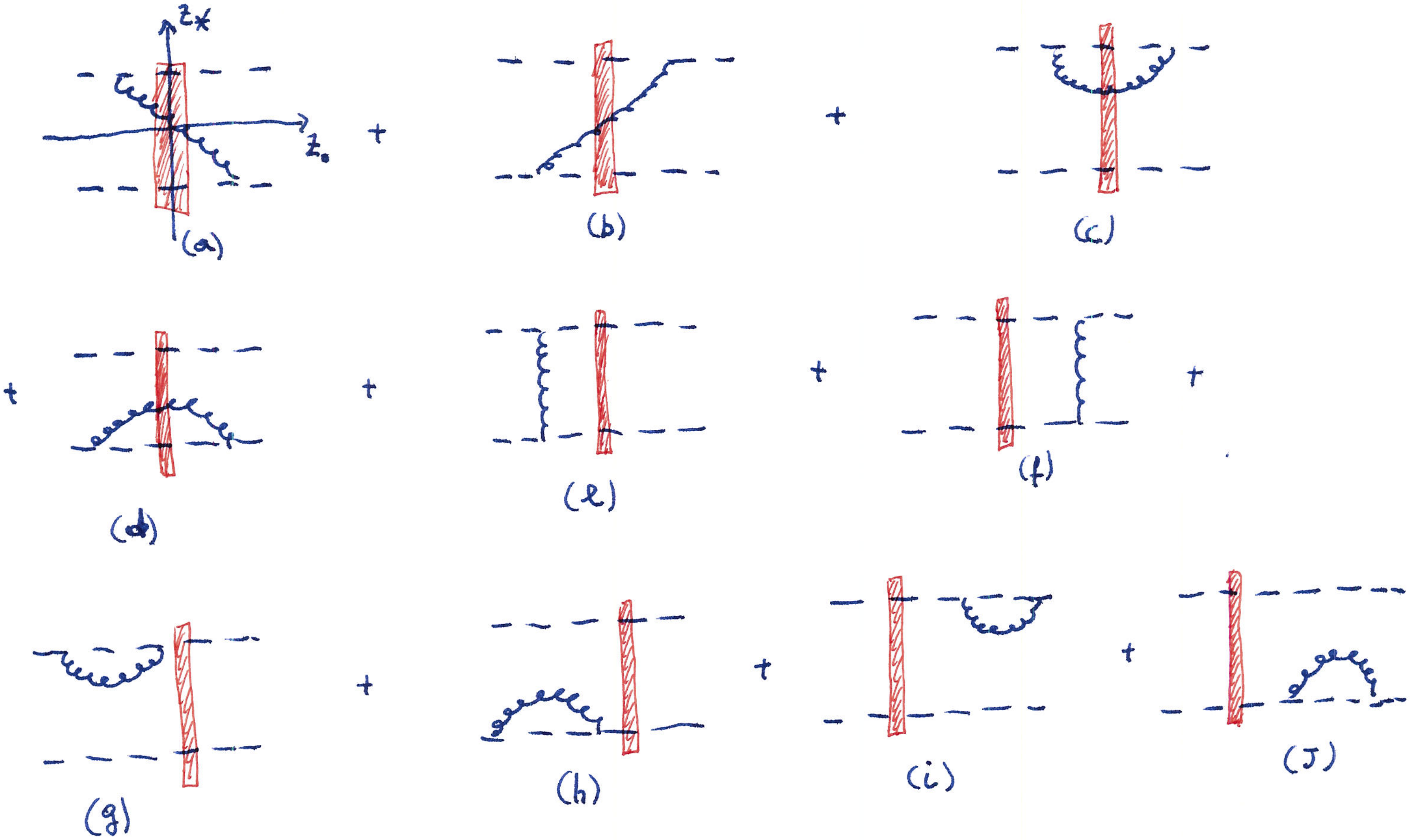


K is the result of the integration over the quantum fluctuations and is the Kernel of the evolution equation.

- Split all the fields into quantum, having rapidity between η_1 and η_2 and classical, having rapidity less than η_2

$$U_{z_1} = P e^{ig\int du\, A_\bullet(p_1 u + z_{1\perp})} \longrightarrow P e^{ig\int du\left(A^q_\bullet(p_1 u + z_{1\perp}) + A^c_\bullet(p_1 u + z_{1\perp})\right)}$$

In the spectator frame the classical field reduces to a shock wave. We calculate perturbatively the contribution of the quantum fluctuation and we obtain the following set of diagrams



diagrams (a) - (d) real diagrams ; (e) - (J) virtual diagrams

Gluon propagator in the shock wave in the axial gauge $\quad A_\mu p_2^\mu = A_* = 0$

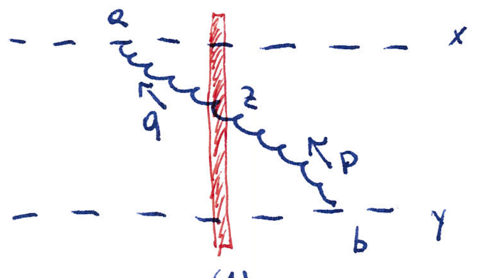
$$\langle A^a_\mu(x)\, A^b_\nu(y)\rangle_A = \theta(x_* y_*)\,\langle x|\,\frac{d_{\mu\nu}\,\delta^{ab}}{i(p^2+i\epsilon)}\,|y\rangle +$$

$$+\,\frac{s}{2}\int d^4z\,\delta(z_*)\,\langle x|\,\frac{d_{\mu\mu'}(p)}{p^2+i\epsilon}\,|z\rangle\Big(2\alpha\,\theta(\alpha)\,g^{\mu'\nu'}\,U_{z_\perp} - 2\alpha\,\theta(-\alpha)\,g^{\mu'\nu'}\,U^\dagger_{z_\perp}\Big)^{ab}\langle z|\,\frac{d_{\nu'\nu}(q)}{q^2+i\epsilon}\,|y\rangle$$

$$d_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_{2\nu} + p_\nu p_{1\mu}}{p\cdot p_2}$$

U_z^{ab} adjoint rep. Wilson line

$$\boxed{d^n p \equiv \frac{d^n p}{(2\pi)^n}}$$



(1)

$$\boxed{\begin{aligned} x_\perp^\mu &= (0, x^1, x^2, 0) \\ x_\perp^\mu x_{\mu\perp} &= -x_\perp^2 = -(x,x)_\perp \\ x^i x^i &= x^1 x^1 + x^2 x^2 = (x,x)_\perp \end{aligned}}$$

U_x^c classical field Wilson line

$$\langle \mathrm{tr}\{U_x U_y^\dagger\}\rangle_A^{\text{Fig 1}} = U_x^c t^a U_y^{c\dagger} t^b \; g^2 \int_{-\infty}^{0} du \int_{0}^{+\infty} dv \,.$$

$$\cdot \frac{s^2}{4}\int d\alpha\, d\beta\, d\beta' \; e^{-i\beta v \frac{s}{2}} e^{i\beta' u \frac{s}{2}} \; \langle y_\perp | \frac{d_{\circ\mu}}{\alpha\beta s - p_\perp^2 + i\epsilon} 2\alpha\theta(\alpha) U^{c\,ba} \frac{d^\mu_\bullet}{\alpha\beta' s - q_\perp^2 + i\epsilon} | x_\perp \rangle$$

$$= -\frac{\alpha_s}{2\pi^2}\int d^2z \left[\mathrm{tr}\{U_x^c U_z^{c\dagger}\}\,\mathrm{tr}\{U_z^c U_y^{c\dagger}\} - \frac{1}{N_c}\mathrm{tr}\{U_x^c U_y^{c\dagger}\}\right] \frac{(x-z, y-z)_\perp}{(x-z)_\perp^2 (y-z)_\perp^2} \int_0^{+\infty} \frac{d\alpha}{\alpha}$$

$\Rightarrow$ Need regularization : Rigid cut-off (is an option)

$$U_x^\eta = P e^{ig\int_{-\infty}^{+\infty} du\, A_\bullet^\eta(up_1 + x_\perp)} \quad ; \quad A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\,\theta(e^\eta - |\alpha_k|)\, e^{-ik\cdot x} A_\mu(k)$$

$$= -\frac{\alpha_s}{2\pi^2}(\eta_1 - \eta_2)\int d^2z \left[\mathrm{tr}\{U_x U_z^\dagger\}\,\mathrm{tr}\{U_z U_y^\dagger\} - \frac{1}{N_c}\mathrm{tr}\{U_x U_y^\dagger\}\right]^{\eta_2} \frac{(x-z, y-z)}{(x-z)_\perp^2 (y-z)_\perp^2}$$

$$\eta_1 - \eta_2 = \Delta\eta$$

$\Rightarrow$

$$\langle \mathrm{tr}\{U_x U_y^\dagger\}\rangle^{\eta_1}_{\text{Fig 1}} = -\frac{\alpha_s}{2\pi^2}\Delta\eta \int d^2 z_\perp \left\langle \left[\mathrm{tr}\{U_x U_z^\dagger\}\,\mathrm{tr}\{U_z U_y^\dagger\} - \frac{1}{N_c}\mathrm{tr}\{U_x U_y^\dagger\}\right]\right\rangle^{\eta_2} \frac{(x-z, y-z)}{(x-z)_\perp^2 (y-z)_\perp^2}$$

sum all diagrams we get

$$\langle \mathrm{tr}\{U_x U_y^\dagger\}\rangle^{\eta_1} = \frac{\alpha_s \Delta\eta}{2\pi^2}\int d^2 z_\perp \frac{(x-y)_\perp^2}{(x-z)_\perp^2 (y-z)_\perp^2}\left\langle\left[\mathrm{tr}\{U_x U_z^\dagger\}\,\mathrm{tr}\{U_z U_y^\dagger\} - N_c\,\mathrm{tr}\{U_x U_y^\dagger\}\right]\right\rangle^{\eta_2}$$

$\Rightarrow$

$$\frac{d}{d\eta}\mathrm{tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2 z_\perp \frac{(x-y)_\perp^2}{(x-z)_\perp^2 (y-z)_\perp^2}\left[\mathrm{tr}\{U_x U_z^\dagger\}\,\mathrm{tr}\{U_z U_y^\dagger\} - N_c\,\mathrm{tr}\{U_x U_y^\dagger\}\right]$$

introduce $\quad \mathcal{U}(x_\perp, y_\perp) = 1 - \frac{1}{N_c}\mathrm{tr}\{U(x_\perp)\, U^\dagger(y_\perp)\}$

$$\frac{d}{d\eta}\mathcal{U}(x_\perp, y_\perp) = \frac{\alpha_s N_c}{2\pi^2}\int d^2 z \frac{(x-y)_\perp^2}{(x-z)_\perp^2 (y-z)_\perp^2}\left[\mathcal{U}(x_\perp, z_\perp) + \mathcal{U}(y_\perp, z_\perp) - \mathcal{U}(x_\perp, y_\perp) - \mathcal{U}(x_\perp, z_\perp)\,\mathcal{U}(z_\perp, y_\perp)\right]$$

B-K equation

Next-to-leading order evolution equation

Formally we may write

$$\frac{d}{d\eta}\mathrm{tr}\{U_x U_y^\dagger\} = \alpha_s K_{LO}\left(\mathrm{tr}\{U_x U_z^\dagger\}\mathrm{tr}\{U_z U_y^\dagger\} - N_c \mathrm{tr}\{U_x U_y^\dagger\}\right) + \alpha_s^2 K_{NLO}\left(\mathrm{tr}\{U_x \cdots\} \cdots\right)$$

we would like to calculate K_{NLO} $\Rightarrow$

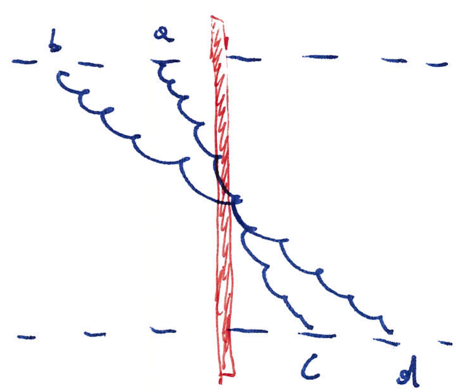
$$K_{NLO}\left(\mathrm{tr}\{U_x \cdots\}\cdots\right) = \frac{d}{d\eta}\mathrm{tr}\{U_x U_y^\dagger\} - \alpha_s K_{LO}\left(\mathrm{tr}\{U_x \cdots\}\cdots\right)$$

$\Rightarrow$ evaluate matrix elements in the external field of a shock wave

- we calculate the right-hand side to obtain the left-hand side

$$\Rightarrow \quad \frac{d}{d\eta}\langle \mathrm{tr}\{U_x U_y^\dagger\}\rangle_{NLO} - \alpha_s K_{LO}\left(\langle \mathrm{tr}\{U_x U_z^\dagger\}\mathrm{tr}\{U_z U_y^\dagger\} - N_c \mathrm{tr}\{U_x U_y^\dagger\}\rangle\right)_{LO}$$

Example of diagram at NLO



NLO kernel

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} &= \frac{\alpha_s}{2\pi^2}\int d^2z\, \Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}] \\
&\times \Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
&-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
&+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\} \\
&-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
&+[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)] \\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)
\end{aligned}
$$

Our result Agrees with NLO BFKL

(Comparing the eigenvalue of the forward kernel)

It respects unitarity