# ELASTIC SHAPE ANALYSIS OF EUCLIDEAN CURVES

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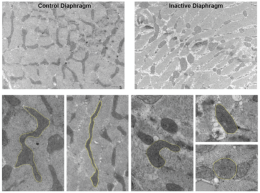
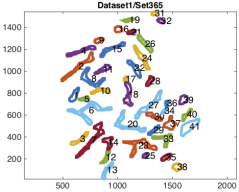
# Outline

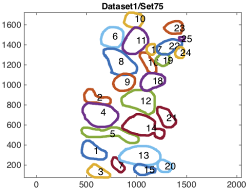
# Outline

# Problem Motivation

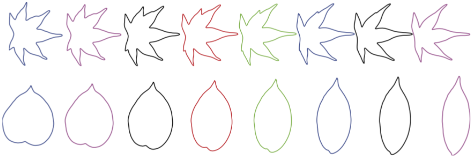
- Mitochondria contours – study of shapes.



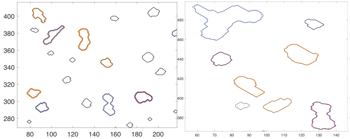




- Leaves

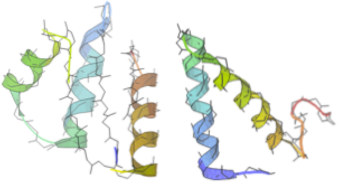


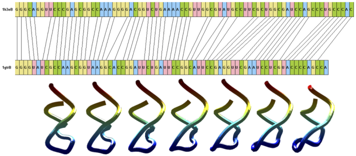
# Problem Motivation

- Nanoparticles:




- Proteins, RNAs – Structure Analysis





# Objects of Interest

- Assume all the objects have the same topology, as described below.
- Euclidean Curves: They are all maps of the type: $f : D \to \mathbb{R}^k$, where $D$ is a one-dimensional compact space. Examples:
  - $D = [0, 1]$: $f$ can be open or closed curve
  - $D = \mathbb{S}^1$: $f$ is called a closed curve
- Curves on Manifolds: They are all maps of the type: $f : D \to M$, where $D$ is a one-dimensional compact space. Examples:
  - $D = [0, 1]$: $f$ is called an open curve
  - $D = \mathbb{S}^1$: $f$ is called a closed curve

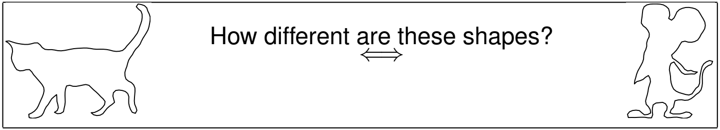
  Often call them trajectories on manifolds.

# Outline

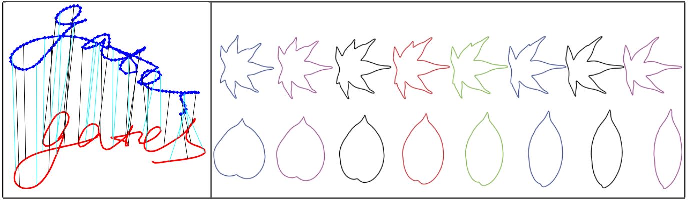
# Specific Goals in Shape Analysis

Shape Analysis: A set of theoretical and computational tools that can provide:

- Shape Metric: Quantify differences in any two given shapes.

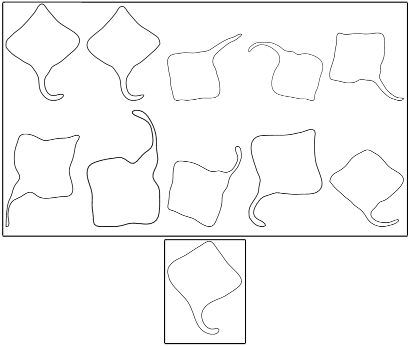



- Registration: Given any two objects find a mapping that assigns each point on an object to a unique point on another object
- Shape Deformation/Geodesic: How to optimally deform one shape into another.



# Shape Analysis

- Shape summary: Compute sample mean, sample covariance, PCA, and principal modes of shape variability.



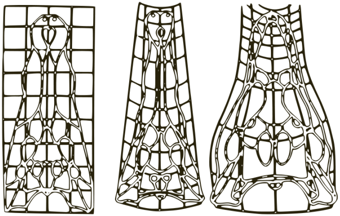
- Shape model and testing: Develop statistical models and perform hypothesis testing.
- Related tools: ANOVA, two-sample test, $k$-sample test, etc.
- Clustering and Classification: Unsupervised and supervised classification of shapes.
- Shape Regression:

# Shape Analysis

- Much older and richer area, with ideas from many perspectives.
- Generally interested in quantifying differences in shapes of objects.
  Kendall: *Shape is a property left after removing shape preserving transformations*.
- Historically statistical shape analysis is restricted to discrete data; each object is represented by a set of points or landmarks.
- Current interest lies in considering continuous objects (examples later). This includes curves and surfaces. These representations can be viewed as functions.
- Functions have shapes and shapes are represented by functions. FDA and shape analysis are quite similar in challenges and solutions.

# On Growth and Form

D’Arcy Thompson – 1905



Crocodilus porosus  C. americanus  Notosuchus terrestris

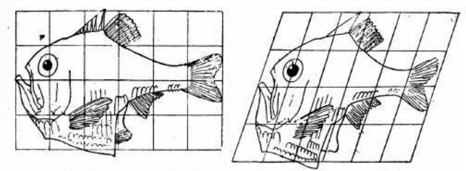
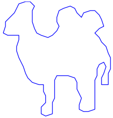
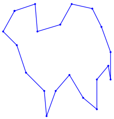


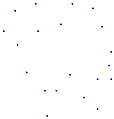
Fig. 517. *Argyropelecus Olfersi.*  Fig. 518. *Sternoptyx diaphana.*

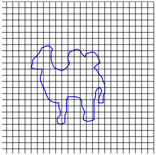
Figure: The top example studies variations in shapes of crocodilian skulls, while the bottom example compares the shape of an *Argyropelecus olfersi* with that of a *Sternoptyx diaphana*. (Data courtesy of Wikipedia Commons.)

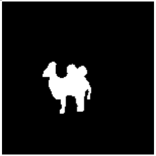
# Shape Representations

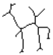

Curves

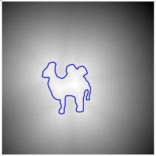

Ordered Samples

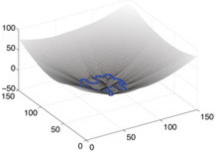

Point Cloud


Deformable Grid


Binary Image


Medial Axis


Signed-Distance



Level Set

# Unlabeled Point Sets

Iterated Closest Point (ICP):

$$\text{RMSD} = \min_{O \in SO(2), \rho \in \mathbb{R}_+, T \in \mathbb{R}^2, \varsigma \in \Sigma} \sum_{i=1}^{k} \|(T + \rho O x_i) - y_{\varsigma(i)}\|^2 .$$

- Translation: $T^* = \frac{1}{k} \sum_{i=1}^{k} y_{\varsigma(i)} - \frac{1}{k} \sum_{i=1}^{k} x_i$.
- Rotation: Compute $A = \sum_{i=1}^{k} y_{\varsigma(i)} x_i^T$ and set

$$O^* = \begin{cases} UV^T & \text{if } \det(A) > 0 \\ U \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} V^T & \text{otherwise.} \end{cases}$$

- Scale: $\rho^* = \frac{\sum_{i=1}^{k} \langle y_{\varsigma(i)}, x_{i,t} \rangle}{\sum_{i=1}^{k} \langle x_{i,t}, x_{i,t} \rangle}$.
- Registration: Nearest neighbor, assignment problem, etc:

$$\varsigma^* = \operatorname*{argmin}_{\varsigma \in \Sigma} \sum_{i=1}^{k} \|y_{\varsigma(i)} - x_{i,t}\|^2 ,$$

# ICP Examples

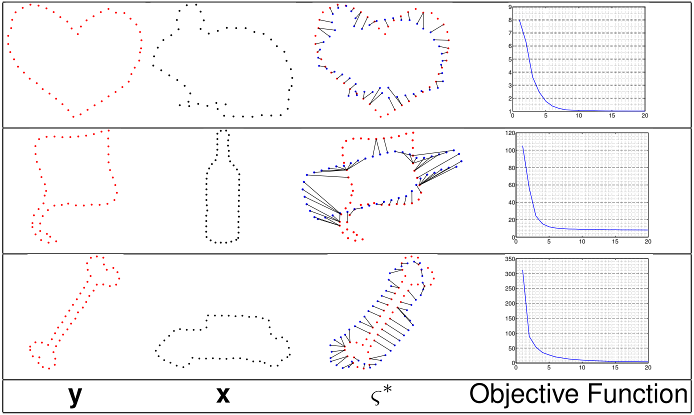



Figure: Examples of matching point clouds using ICP algorithm.

# ICP Examples

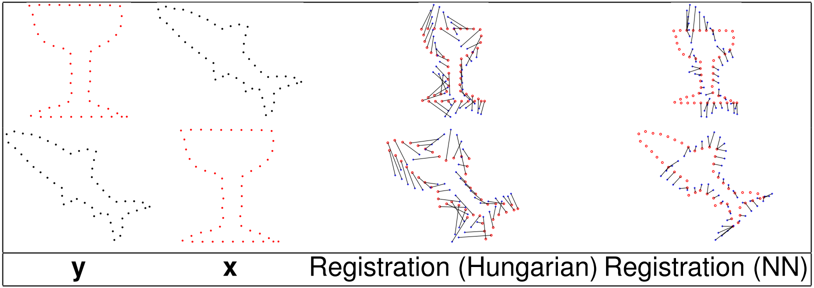



Figure: Different optimal registrations of two points sets using the nearest neighbor (NN) algorithm and the Hungarian algorithm

# Active Shape Models

- Consider the set of $n$ landmarks on an object as an $n \times 2$ matrix.
- Center the configuration by subtracting the mean.
- Rescale each configuration by dividing by its norm.
- Perform rotational alignment and compute straight line geodesics.

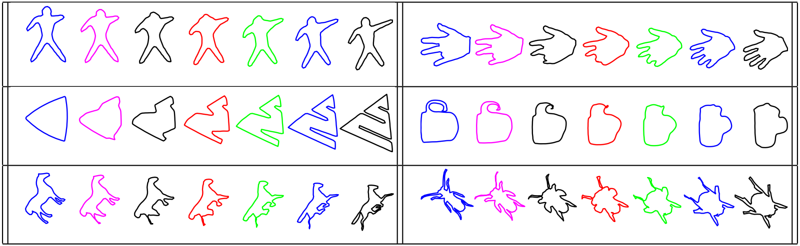


Figure: Examples of geodesic paths between same shapes using ASM.

# Kendall’s shape analysis

- Same thing except respect the geometry of the underlying space.

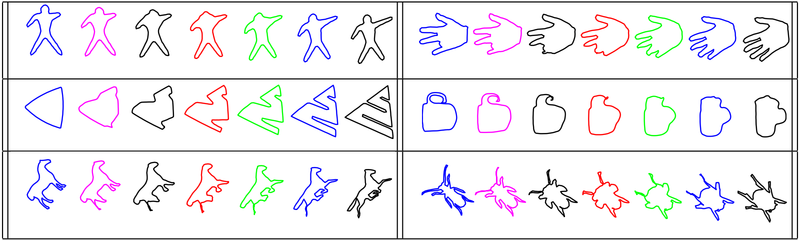
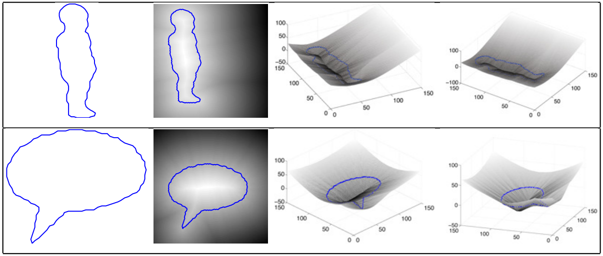


Figure: Examples of geodesic paths between same shapes using ASM.

# Signed-Distance Function

- Signed-Distance functions



- Difficult to find a geodesic path in the space of signed distance functions.
- Difficult to be invariance to rotation.
- Registration is pre-determined.

# Outline

# Registration Problem

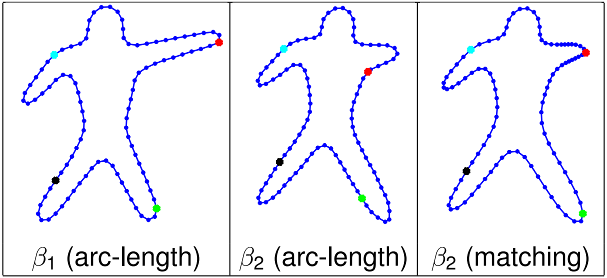



Figure: Registration of points across two curves using the arc-length and a convenient non-uniform sampling. Non-uniform sampling allows a better matching of features between $\beta_1$ and $\beta_2$.

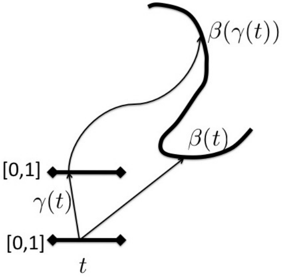
# Elastic Shape Analysis

**Elastic Shape Analysis**

Perform registration and shape comparison (analysis) simultaneously.

# Mathematical Representations of Curves

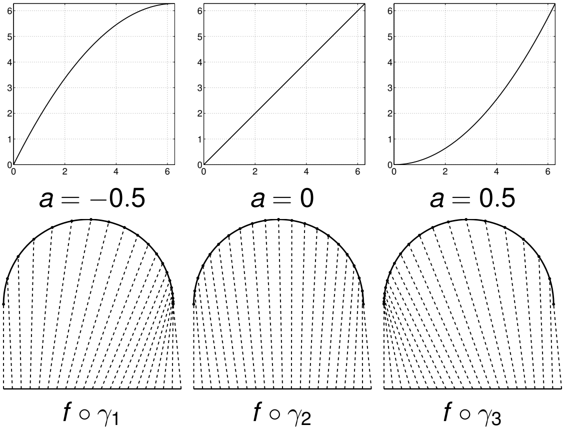
- Parametrized curves – $f : [0,1] \to \mathbb{R}^2$, $\mathbb{S}^1 \to \mathbb{R}^2$.




- Let $\Gamma$ be the set of all diffeomorphisms of $[0,1]$ that preserve the boundaries. Elements $\gamma \in \Gamma$, plays the role of a re-parameterization function.
- For any curve $f : [0,1] \to \mathbb{R}^2$, and $\gamma \in \Gamma$, the composition $f \circ \gamma$ is a re-parameterization of $f$.
- $\Gamma$ is a group (with composition as group operation), and $f \mapsto (f,\gamma) = f \circ \gamma$ defines a group action on the space of curves.

# Example: Re-Parameterization

Example: $\gamma_a(t) = t + at(1-t), \quad -1 < a < 1.$



# Shape-Preserving Transformations

Following group actions are shape preserving:

- Translation: For any $x \in \mathbb{R}^2$, the $f(t) \mapsto x + f(t)$ denotes a translation of $f$.
- Rotation: For any $O \in SO(2)$, the $f(t) \mapsto Of(t)$ denotes a rotation of $f$.
- Scaling: For any $a \in \mathbb{R}_+$, the $f(t) \mapsto af(t)$ denotes the translation of $f$.
- Re-parameterization: For any $\gamma \in \Gamma$, $f(t) \mapsto f(\gamma(t))$ is a re-paramaterization of $f$.

We want shape metrics and shape analysis to be invariant to these actions. For instance, if $d_s$ is a shape metric, then we want:

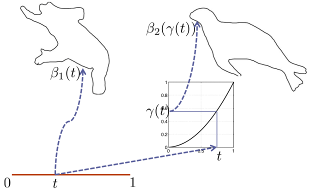
$$d_s(f_1, f_2) = d_s(aO(f_1 \circ \gamma) + x, f_2), \ \ \forall a \in \mathbb{R}_+, O \in SO(2), \gamma \in \Gamma, x \in \mathbb{R}^2$$

These transformations are considered nuisance in shape analysis.

# Registration Through Re-Parametrizations

Re-parameterization is not entirely a nuisance transformation. It is useful in solving the registration problem.

- Take two parameterized curves $f_1, f_2 : [0, 1] \to \mathbb{R}^2$.
- For any $t$, the point $f_1(t)$ on the first curve is said to be registered to the point $f_2(t)$ on the second curve.
- We can change the registration by re-parametrizing the curves.
- If we re-parameterize $f_2$ by $\gamma$, then the new registration is $f_1(t) \leftrightarrow f_2(\gamma(t))$.



# Metrics for Registration/Shape Comparsions

- We need an objective function to define optimality of registration.
- The $\mathbb{L}^2$ norm seems like a natural choice but it suffers from the pinching effect.

$$\inf_{\gamma \in \Gamma} \|f_1 - f_2 \circ \gamma\|$$

- As earlier, the main problem is $\|f\| \neq \|f \circ \gamma\|$, in general.
- We need a metric that satisfies:

$$d(f_1, f_2) = d(f_1 \circ \gamma_1, f_2 \circ \gamma_2), \quad \forall \gamma_1, \gamma_2 \in \Gamma \ .$$

- We will define an elastic shape metric (start with a Riemannian metric and then derive distance under that metric) that satisfies this property.
- This metric will be too complex to use directly, so we will simplify it using a square-root transformation.

# Outline

# Elastic Riemannian Metric

- Let $f : [0,1] \to \mathbb{R}^n$ be a Euclidean curve. $\dot{f}(t)$ is the velocity vector at $f(t)$.
  - $r(t) = |\dot{f}(t)|$ is the speed function, and
  - $\Theta(t) = \frac{\dot{f}(t)}{r(t)}$ is the direction vector.

  We represent a curve by the pair $(r, \Theta)$.
- For a re-parameterized curve $f \circ \gamma$, the representation is given by $((r \circ \gamma)\dot{\gamma}, \Theta \circ \gamma)$.
- Elastic Riemannian Metric for curves: for any $a, b$,

$$\begin{aligned}\langle(\delta r_1, \delta\Theta_1), (\delta r_2, \delta\Theta_2)\rangle_{(r,\Theta)} &= a^2 \int_0^1 \delta r_1(t)\delta r_2(t)\frac{1}{r(t)}\,dt \\ &+ b^2 \int_0^1 \delta\Theta_1(t)\delta\Theta_2(t)r(t)\,dt.\end{aligned}$$

- This metric is invariant to re-parameterization of $f$:

$$\langle(\delta((r_1 \circ \gamma)\dot{\gamma}), \delta(\Theta_1 \circ \gamma)), (\delta((r_2 \circ \gamma)\dot{\gamma}), \delta(\Theta_2 \circ \gamma))\rangle_{(((r\circ\gamma)\dot{\gamma}),(\Theta\circ\gamma))}$$
$$= \langle(\delta r_1, \delta\Theta_1), (\delta r_2, \delta\Theta_2)\rangle_{(r,\Theta)}$$

# SRVF Representation for Curves

- Define the square-root velocity function (SRVF): $q(t) \equiv \frac{\dot{f}(t)}{\sqrt{|\dot{f}(t)|}} = \sqrt{r(t)}\Theta(t)$.
- Computing variation on both sides, we get:

$$\delta q = \frac{1}{2\sqrt{r(t)}}\delta r(t)\Theta(t) + \sqrt{r(t)}\delta\Theta(t) \ .$$

- Taking standard $\mathbb{L}^2$ inner product between two such variations:

$$\langle \delta q_1, \delta q_2 \rangle = \frac{1}{4}\int_0^1 \delta r_1(t)\delta r_2(t)\frac{1}{r(t)}dt + \int_0^1 \langle \delta\Theta_1(t), \delta\Theta_2(t) \rangle \, r(t)dt \ .$$

  Use $\langle \Theta(t), \delta\Theta_i(t) \rangle = 0$.
- This is equal to the elastic Riemannian metric for $a = 1/2$ and $b = 1$. Thus, the mapping $f \mapsto q$ transforms the elastic Riemannian metric into the $\mathbb{L}^2$ metric for these weights.
- The geodesic distance between any $f_1$ and $f_2$ under the elastic Riemannian metric (for $a = 1/2$ and $b = 1$) is simply $\|q_1 - q_2\|$.

# SRVF Representation ....

- We use SRVF $q$ for analyzing shape of a curve $f$.
- The SRVF of $(f \circ \gamma)$ is $(q \circ \gamma)\sqrt{\dot{\gamma}}$. Just by chain rule. We will denote $(q, \gamma) = (q \circ \gamma)\sqrt{\dot{\gamma}}$.
  Commutative Diagram:

$$\begin{array}{ccc} f & \xrightarrow{\text{SRVF}} & q \\ \text{Group action by } \Gamma \downarrow & & \downarrow \text{Different Group action by } \Gamma \\ (f \circ \gamma) & \xrightarrow[\text{SRVF}]{} & (q, \gamma) \end{array}$$

- *Lemma*: The chosen distance satisfies:
  $d_{FR}(f_1, f_2) = d_{FR}(f_1 \circ \gamma, f_2 \circ \gamma)$
  We need to show that $\|(q_1 \circ \gamma)\sqrt{\dot{\gamma}} - (q_2 \circ \gamma)\sqrt{\dot{\gamma}}\| = \|q_1 - q_2\|$.

$$\begin{aligned} \|(q_1, \gamma) - (q_2, \gamma)\|^2 &= \int_0^1 (q_1(\gamma(t))\sqrt{\dot{\gamma}(t)} - q_2(\gamma(t))\sqrt{\dot{\gamma}(t)})^2 dt \\ &= \int_0^1 (q_1(\gamma(t)) - q_2(\gamma(t)))^2 \dot{\gamma}(t) dt = \|q_1 - q_2\|^2 . \square \end{aligned}$$

# Shape Analysis Using SRVFs

- Checking all nuisance transformations:
  1. Translation: SRVF $q$ for a curve $f$ is invariant to its translation !
  2. Scaling: We can rescale all the curves to be of unit length, to get rid of the scale variability. It turns out that $\|q\| = L[f]$. So, if $L[f] = 1$, then the corresponding SRVF $q$ is an element of a unit sphere $\mathbb{S}_\infty$.
  3. Re-parameterization and rotations we can't remove by any such standardization. However, we have the nice property:

$$\|q_1 - q_2\| = \|Oq_1 - Oq_2\| = \|(q_1, \gamma) - (q_2, \gamma)\| \ .$$

- We use the notion of equivalence classes, or orbits, to reconcile the remaining two transformation. For any curve $f$, and its SRVF $q$, we its equivalence class to be:

$$[q] = \{O(q, \gamma) | O \in SO(n), \gamma \in \Gamma\} \ .$$

  This set represents SRVFS of all possible rotations and re-parameterizations of $f$. Each equivalence class represents a shape.

# Shape Metric

- $\mathbb{S}_\infty \subset \mathbb{L}^2$ is called the pre-shape space.
- The set of all equivalence classes is a quotient space $\mathbb{L}^2/(SO(n) \times \Gamma)$. It is called the shape space.
- The distance between any two curves in the pre-shape space is $\cos^{-1}(\langle q_1, q_2 \rangle)$.
- The distance in the shape space, called the shape metric, is given by:

$$d_s([q_1],[q_2]) = \inf_{(O,\gamma)\in SO(n)\times\Gamma} \cos^{-1}(\langle q_1, O(q_2,\gamma)\rangle) \; .$$

  This include rotational alignment and non-rigid registration of the two curves.
- Given optimal parameters $O^*, \gamma^*$, the shortest path or a geodesics is simply:

$$\alpha(\tau) = \frac{1}{\sin(\vartheta)}(\sin(\vartheta(1-t))q_1 + \sin(\vartheta t)q_2^*), \;\; \cos(\vartheta) = \langle q_1, q_2^* \rangle \; ,$$

  where $q_2^* = O^*(q_2, \gamma^*)$.

# Shape Metric

- So far we have developed a technique for computing geodesics and geodesic distances in shape space of curves.
- Suppose we are interested in only closed curves.
- The SRVF $q$ of a closed curve $f$ satisfies an additional condition:

$$f(0) = f(1) \Leftrightarrow \int_0^1 q(t)|q(t)|dt = 0 \ .$$

- So we are now interested in the pre-shape space:

$$\mathcal{C} = \{q \in \mathbb{S}_\infty | \int_0^1 q(t)|q(t)|dt = 0\} \subset \mathbb{S}_\infty \ .$$

  The geodesics here are no longer arcs on great circles. We don't know have analytical expressions for these geodesics or geodesic distances.
- We have developed a numerical technique called path straightening for finding geodesics on $\mathcal{C}$.

# Outline

# Path Straightening Method: Theoretical Background

- The goal is to find a (locally) shortest path between two points $p$ and $q$ on a Riemannian manifold $M$.

$$\hat{\alpha} = \arg \inf_{\alpha:[0,1]\to M, \alpha(0)=p, \alpha(1)=q} \int_0^1 \sqrt{\langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle_{\alpha(t)}} dt$$

$$= \arg \inf_{\alpha:[0,1]\to M, \alpha(0)=p, \alpha(1)=q} \int_0^1 \langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle_{\alpha(t)} \, dt$$

- Define $E[\alpha] = \int_0^1 \langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle_{\alpha(t)} \, dt$.
- The set of all paths is:

$$\mathcal{A} = \{\alpha : [0,1] \to M |\ \alpha \text{ is differentiable and } \dot{\alpha} \in \mathbb{L}^2([0,1], M)\} ,$$

- The subset of paths with desired boundary conditions:

$$\mathcal{A}_0 = \{\alpha \in \mathcal{A} | \alpha(0) = p_1 \text{ and } \alpha(1) = p_2\} .$$

# Path Straightening Method: Theoretical Background

- The tangent spaces to these sets of paths:

$$T_\alpha(\mathcal{A}) = \left\{ w : [0,1] \to TM \middle| \frac{Dw}{d\tau} \in \mathbb{L}^2 \text{ and } \forall \tau \in [0,1], w(\tau) \in T_{\alpha(\tau)}(M) \right\},$$

  where $T_{\alpha(\tau)}(M)$ is the tangent space of $M$ at the point $\alpha(\tau) \in M$
- Each element of $T_\alpha(\mathcal{A})$ is a (tangent) vector field along $\alpha$.
- For the set $\mathcal{A}_0$, the tangent space is

$$T_\alpha(\mathcal{A}_0) = \{ w \in T_\alpha(\mathcal{A}) | w(0) = w(1) = 0 \} .$$

- This is a set of vector fields along $\alpha$ that are zero at the boundaries.
- The optimization problem is:

$$\hat{\alpha} = \arg \inf_{\alpha \in \mathcal{A}_0} \int_0^1 \langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle \, dt$$

# Gradient of Energy $E$

**Theorem**

*Let $\alpha : [0,1] \to M$ be a path such that $\alpha(0) = p_1$ and $\alpha(1) = p_2$, i.e. $\alpha \in \mathcal{A}_0$. Then, with respect to the Palais metric:*

1. *The gradient of the energy function $E$ on $\mathcal{A}$ at $\alpha$ is the vector field $u$ along $\alpha$ satisfying $u(0) = 0$ and $\frac{Du}{d\tau} = \frac{d\alpha}{d\tau}$.*
2. *The gradient of the energy function $E$ restricted to $\mathcal{H}_0$ is $w(\tau) = u(\tau) - \tau\tilde{u}(\tau)$, where $u$ is the vector field defined in the previous item, and $\tilde{u}$ is the vector field obtained by parallel translating $u(1)$ backwards along $\alpha$.*

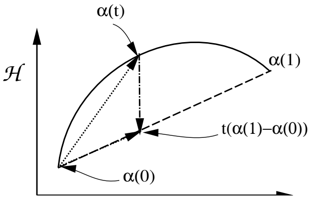
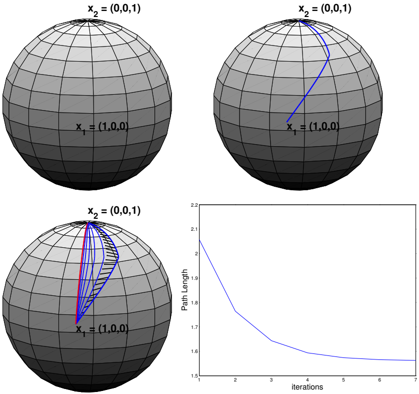



Figure: Illustration of path-straightening update on a curve in $\mathbb{R}^2$.

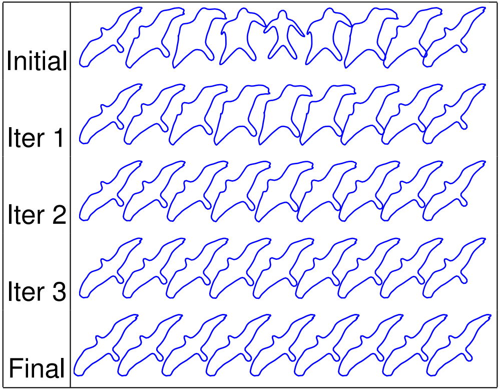
# Path Straightening for a Unit Sphere

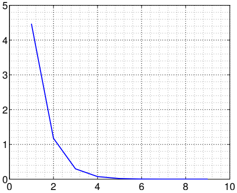
An example of path-straightening method for computing geodesics between two points on $\mathbb{S}^2$.



# Path Straightening in Pre-Shape Space of Closed Curves

Example:



# Path Straightening Examples

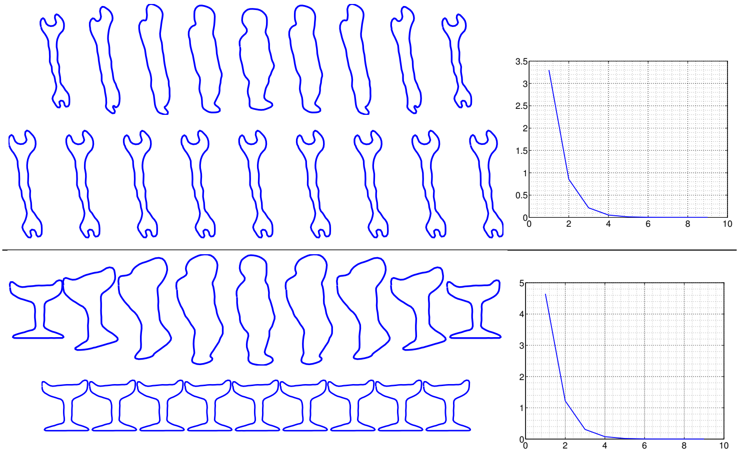



Figure: Illustration of path straightening: each example shows an initial path (top), the final path (bottom left), and the evolution of the path energy $E$ (bottom right).

# Overall Scheme for Computing Geodesics

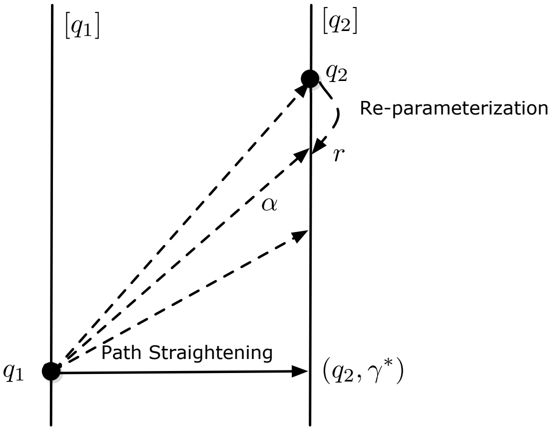
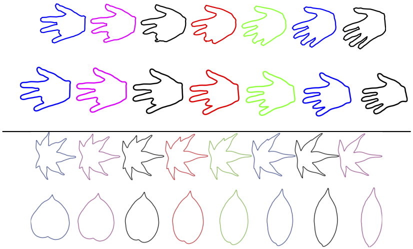
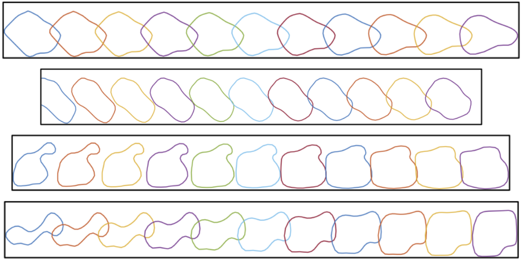



Figure: Gradient-based update of elements in $[q_2]$, while keeping $q_1$ fixed, to find the shortest geodesic between the orbits of $[q_1]$ and $[q_2]$.

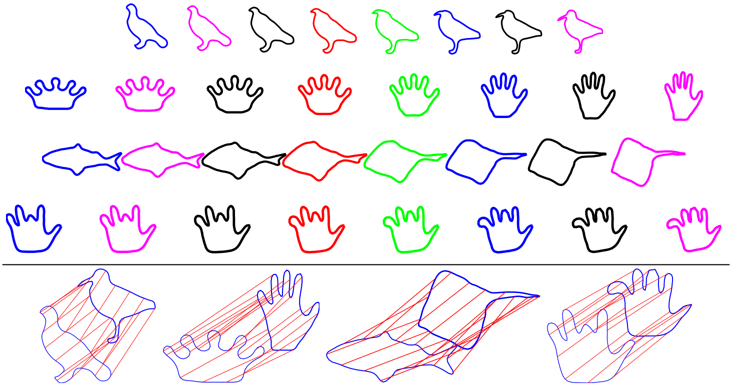
# Elastic Geodesics

- Hand contours/ Leaves/ Nanoparticles





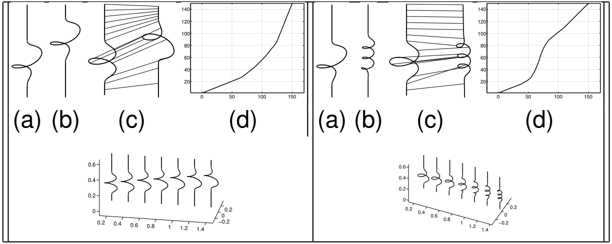
# Elastic Geodesics



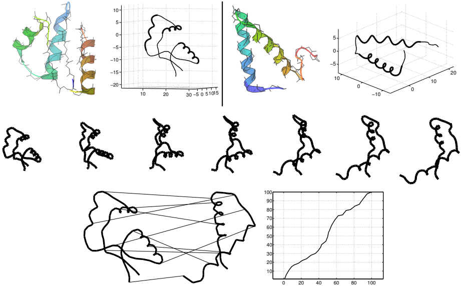
# Elastic Geodesics 3D Curves

All these ideas extend easily to curves in higher dimensions.

- Example 1:

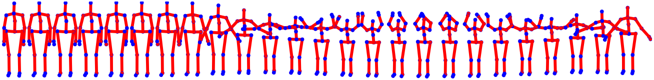



- Example 2:

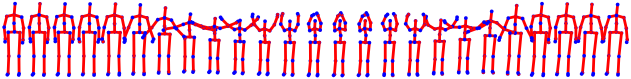


# Elastic Registration of High-Dimensional Curves

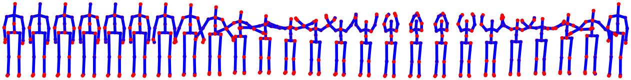
Temporal alignment of human activity data: Two-hand wave



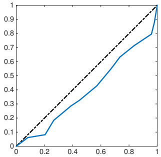
Sequence 1, $f_1$

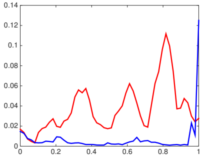


Sequence 2, $f_2$



Sequence 2 re-parameterized, $f_2 \circ \gamma_1^*$

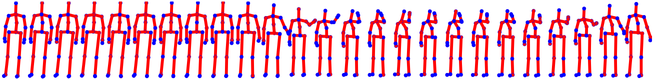


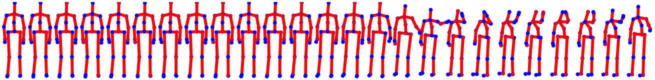

Warping $\gamma^*$ $|q_1(t) - q_2(t)|$, $|q_1(t) - q_2(\gamma^*(t))\sqrt{\dot{\gamma}^*(t)}|$

# Elastic Registration of High-Dimensional Curves

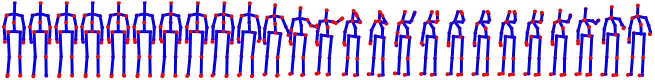
Temporal alignment of human activity data: One-arm wave



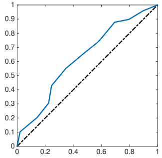
Sequence 1, $f_1$

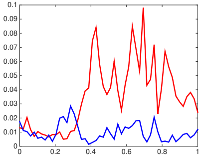


Sequence 2, $f_2$



Sequence 2 re-parameterized, $f_2 \circ \gamma_1^*$



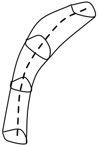
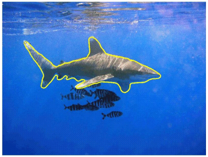
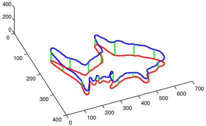
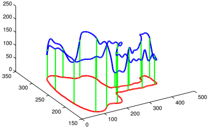




Warping $\gamma^*$

Plots of $|q_1(t) - q_2(t)|$ and $|q_1(t) - q_2(\gamma^*(t))\sqrt{\dot{\gamma}^*(t)}|$

# Outline

# Shape Analysis of Colored Curves

<table>
<tr><td></td><td> 
</td></tr>
<tr><td>Generalized cylinder</td><td>Colored curve</td></tr>
<tr><td></td><td> 

Colored curve</td></tr>
</table>

# Shape Analysis of Colored Curves

- Let the shape coordinate function along a closed curves be given by $\beta_s : D \rightarrow \mathbb{R}^n$ and the auxiliary function be given by $\beta_t : D \rightarrow \mathbb{R}^k$.
- Form a joint shape and texture curve: $\beta(t) = \begin{bmatrix} \beta_s(t) \\ b\beta_t(t) \end{bmatrix} \in \mathbb{R}^{n+k}$. Here $b > 0$ is a parameter introduced to control the influence of the auxiliary function, relative to the shape function.
- Define *augmented pre-space* as:

$$\mathcal{C}_2 = \{q : D \rightarrow \mathbb{R}^{(n+k)} | \int_D \langle q(t), q(t) \rangle \, dt = 1, \ \int_D |q(t)| q(t) \, dt = 0\} .$$

- The rotation group acting on this space is given by

$$\mathcal{R} = \begin{bmatrix} SO(n) & 0 \\ 0 & I_k \end{bmatrix} \subset SO(n+k),$$

  where $I_k$ is the $k \times k$ identity matrix.
- Orbits under the joint action of the rotation and the re-parameterization group:

$$[q] = \{O(q \circ \gamma)\sqrt{\dot{\gamma}} | O \in \mathcal{R}, \gamma \in \tilde{\Gamma}\} .$$

- To compare any two objects, represented by $([q^1], \bar{\beta}_0^1)$ and $([q^2], \bar{\beta}_0^2)$, we use the distance function:

$$d(\beta^1, \beta^2; b) = \left( \sqrt{d_s([q^1], [q^2])^2 + |\bar{\beta}_0^1 - \bar{\beta}_0^2|^2} \right) \quad (1)$$

# Example

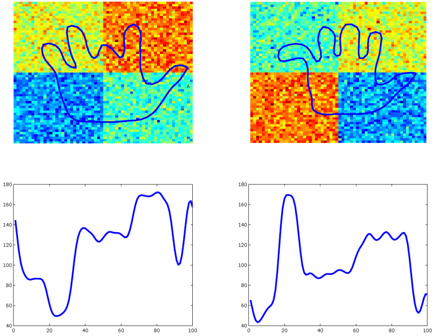
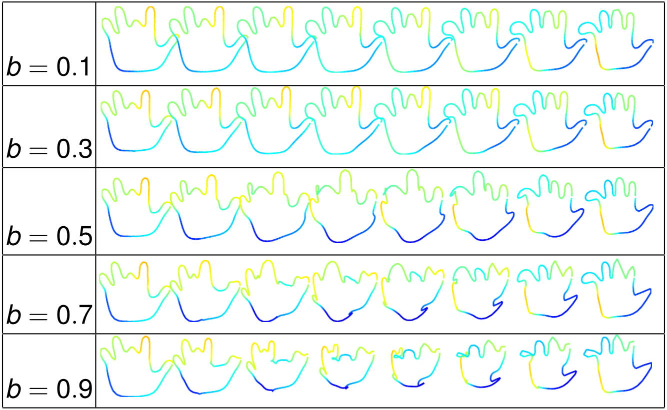



Figure: Top row: two hand shapes immersed in artificial texture. Bottom row: the texture functions along the two curves after smoothing.

# Example

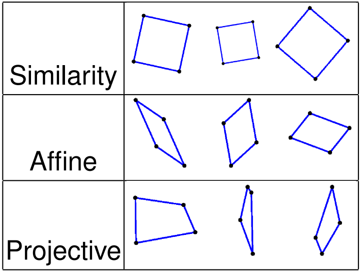


# Outline

# Affine-Invariant Elastic Shape Analysis

- Now we want the shape analysis to be invariant to the action of the affine group.
- The affine group for a plane is the semi-direct product $\mathcal{G}_A \equiv GL(2) \ltimes \mathbb{R}^2$ with the action given by: $\mathcal{G}_A \times \mathbb{R}^2 \to \mathbb{R}^2$

$$((A, b), x) = Ax + b \,.$$




- Let $\beta : \mathbb{S}^1 \to \mathbb{R}^2$ be a planar closed curve. The affine-orbit of $\beta$ is the set

$$[\beta]_A = \left\{ A\beta + b \mid A \in GL(2), b \in \mathbb{R}^2 \right\} .$$

# Affine Standardization of Curves

**Theorem**

*For any non-degenerate $\beta$ there exists a standardized element $\beta^* \in [\beta]_A$, the affine-orbit of $\beta$, that satisfies the following three conditions:*

1. $L_{\beta^*} = 1$,
2. *Centroid of $\beta$,* $C_{\beta^*} = 0$, *and*
3. *Covariance of points along $\beta$,* $\Sigma_{\beta^*} \propto I$.

*Furthermore, for any two curves $\beta_1, \beta_2 \in [\beta]_A$, the corresponding standardized elements, $\beta_1^*$ and $\beta_2^*$, are related by a rotation and re-parameterization, $\beta_2^* = O(\beta_1^* \circ \gamma)$, where $O \in SO(2)$ and $\gamma \in \Gamma$.*

Thus, we can standardize the given affine-transformed curves and apply elastic shape analysis derived earlier.

# Affine Standardization of Curves

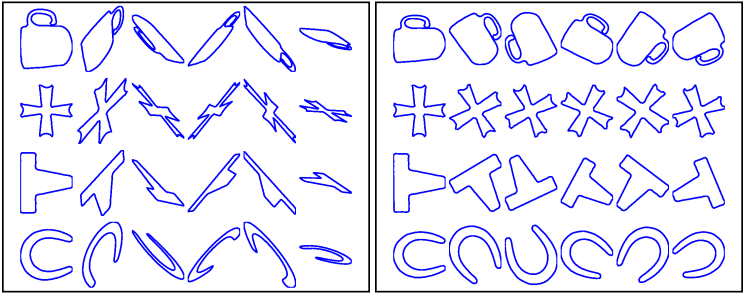


Figure: Affine standardization of curves. The original curves are shown in the left and their standardizations are shown in the right.

# Geodesics Using Path Straightening

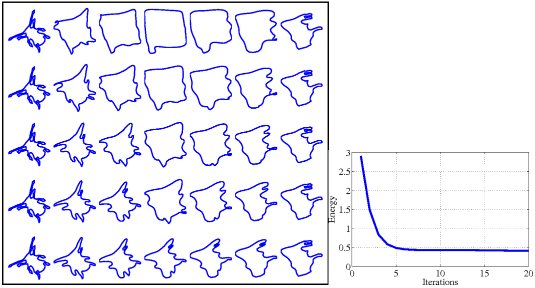



Figure: Path-straightening on affine pre-shape space. The left side shows Iterations of the path-straightening algorithm from top (initial path) to bottom (final path). The right panel shows the corresponding evolution of the path energy.

# De-Standardization of Shapes Along Geodesic

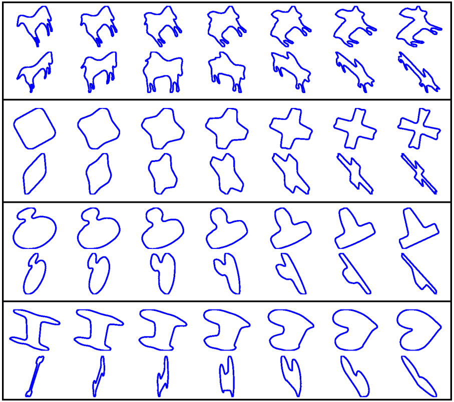


Figure: Each case shows a geodesic in standardized similarity shape space (top row) and its de-standardization (bottom row).

# Summary: Shape Analysis of Curves

- For registration of points across curves one needs an invariant Riemannian metric, leading to an invariant distance.
- This metric is too complex to be useful in practical situations. A square-root transformation, SRVF, converts this metric into a simpler $\mathbb{L}^2$ metric.
- We define quotient spaces of $\mathbb{L}^2$ under shape-preserving transformations, such as the rotation and re-parameterizations.
- All the operations – registration, geodesics, statistical analysis, etc. – take place in the SRVF space. Final solutions are converted back to curve space by inverting SRVFs.

# Outline

# Shape Clustering

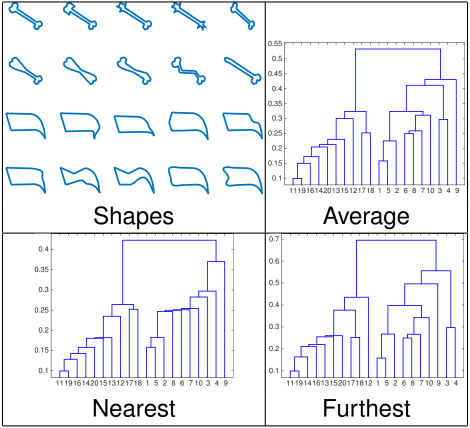



Figure: A set of 20 shapes of the left have been clustered using different linkage criterion: average (top-right), nearest distance (bottom left), and compete or furthest distance (bottom-right).

# Shape Clustering

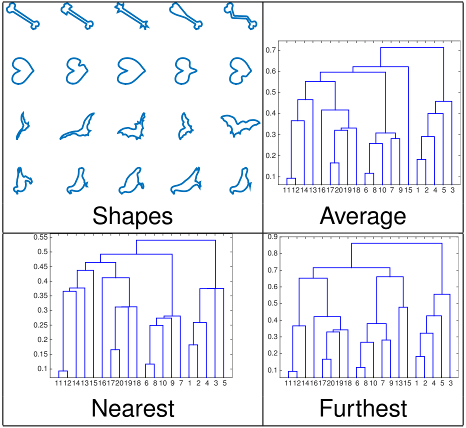
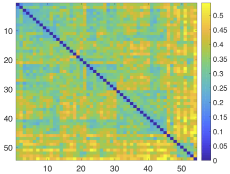
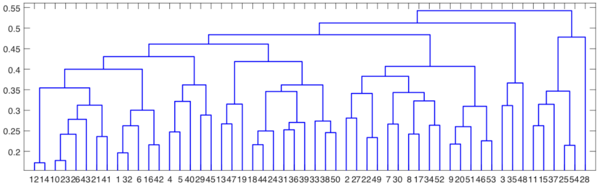
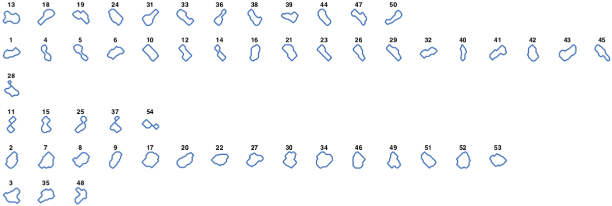



Figure: A set of 20 shapes of the left have been clustered using different linkage criterion: average (top-right), nearest distance (bottom left), and compete or furthest distance (bottom-right).

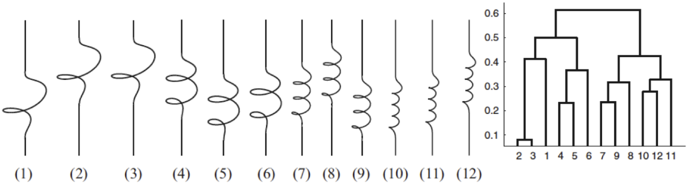
# Shape Clustering



# 3D Shape Clustering



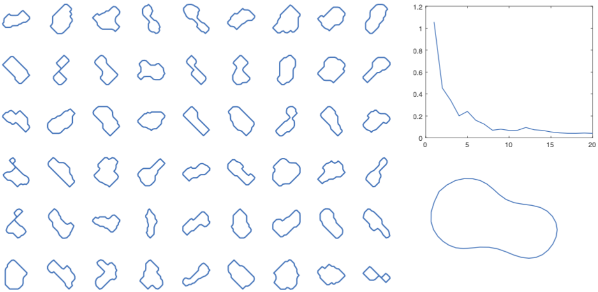
# Outline

# Shape Statistics

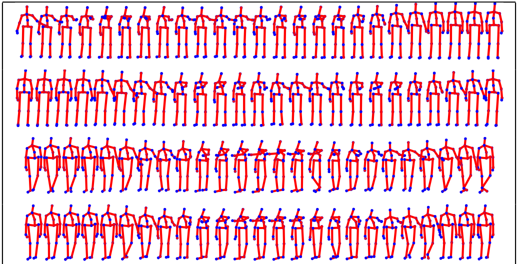
- Sample mean:

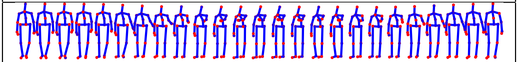
$$\mu_q = \underset{[q]\in\mathcal{S}}{\operatorname{argmin}} \sum_{i=1}^{n} d_s([q],[q_i])^2 ,$$

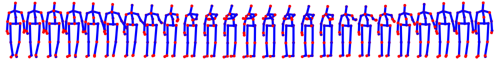
and then, $\mu_q \mapsto \mu$.



# Elastic Averaging of Multiple Shape Sequences

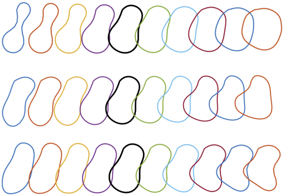

Four of Six Sequences Used in Experiment

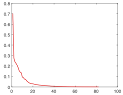

Pre-Alignment Mean


Post-Alignment Mean

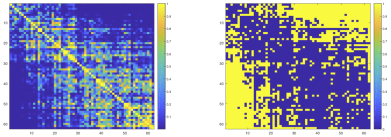
# Shape Statistics

- PCA in the tangent space at the mean





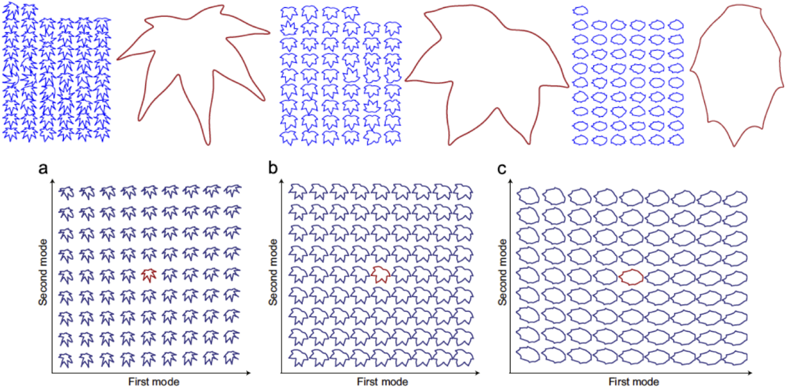

- Testing equality of shape populations across time frames: Truncated Wrapped Normal Distributions




*p* values (left) and binary decisions (right)

The nanoparticle shape populations across frames are increasing different as the frames are further apart in time.

# Leaves Shapes



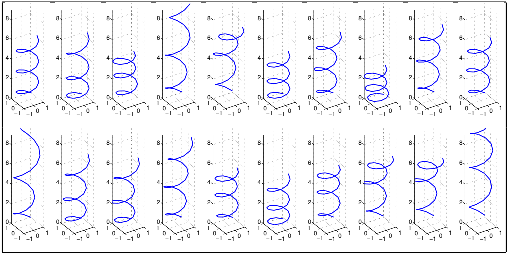
# Leaves Classification

| Methods | Recognition score |
|---|---|
| **SM200** | **99.18** |
| TAR (Mouine et al., 2013a, 2013b) | 90.40 |
| TSL (Mouine et al., 2013a, 2013b) | 95.73 |
| TOA (Mouine et al., 2013a, 2013b) | 95.20 |
| TSLA (Mouine et al., 2013a, 2013b) | 96.53 |
| Shape-Tree (Felzenszwalb and Schwartz, 2007) | 96.28 |
| IDSC + DP (Ling and Jacobs, 2007) | 94.13 |
| SC + DP (Ling and Jacobs, 2007) | 88.12 |
| Fourier descriptors (Ling and Jacobs, 2007) | 89.60 |

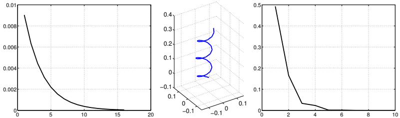
| Method | Score |
|---|---|
| **SM200** (this paper) | **0.953** |
| TAR (Mouine et al., 2013a, 2013b) | 0.636 |
| TSL (Mouine et al., 2013a, 2013b) | 0.757 |
| TOA (Mouine et al., 2013a, 2013b) | 0.780 |
| TSLA (Mouine et al., 2013a, 2013b) | 0.779 |
| IFSC_USP_run2 | 0.402 |
| inria_imedia_plantnet_run1 | 0.464 |
| IFSC_USP_run1 | 0.430 |
| LIRIS_run3 | 0.513 |
| LIRIS_run1 | 0.543 |
| Sabanci-okan-run1 | 0.476 |
| LIRIS_run2 | 0.508 |
| LIRIS_run4 | 0.538 |
| inria_imedia_plantnet_run2 | 0.554 |
| DFH + GP (Yahiaoui et al., 2012) | 0.725 |

# PCA of Curves in $\mathbb{R}^3$

(a) A collection of 20 spiral curves used in this experiment



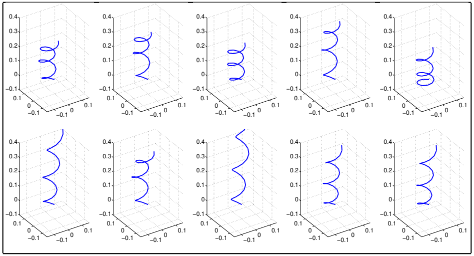
(a)



(b) (c) (d)

(b) the decrease in the norm of the gradient of Karcher variance function during mean estimation, (c) the estimated Karcher mean and (d) the estimated singular values of the covariance matrix.

# PCA of Curves in $\mathbb{R}^3$



Random samples from the estimated wrapped-normal density in the shape space.