SHAPE ANALYSIS OF TRAJECTORIES ON MANIFOLDS

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Outline

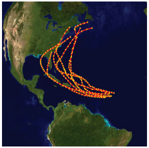
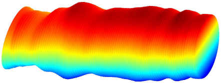
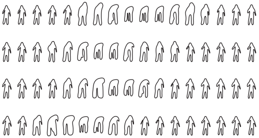
Problem Description

Interested in curves of the type $\alpha : [0, 1] \to M$, where M is a nonlinear Riemannian manifold. Longitudinal data on manifolds.

- Spherical Trajectories: $M = \mathbb{S}^d$ a unit sphere
  Directional data, geographical data.
- Covariance Trajectories: $M = \mathbb{P}$, the set of symmetric, positive definite matrices.
  Brain connectivity data
- Shape Trajectories: $M = \mathcal{S}$ a shape space
  Video data, action recognition.
- Graph Trajectories: $M = \mathcal{G}$ a space of graphs
  social networks, recommender systems.

Some of the Applications

- Activity recognition using video and depth sensing.
- Hurricane trajectories.
- Bird migration data.
- Dynamical functional connectivity analysis
- Biological growth data

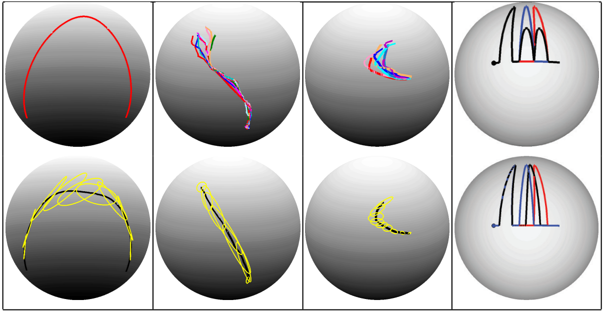
Metrics for Comparing Trajectories

Consider an arbitrary Riemannian manifold M.

- Let d_m be the geodesic distance on M. In order to compare any two trajectories $\alpha_1, \alpha_2 : [0,1] \to M$, one use the metric:

$$d_x(\alpha_1, \alpha_2) = \int_0^1 d_m(\alpha_1(t), \alpha_2(t))dt.$$

- However, the given data may lack temporal registration. We need to register the trajectories.
- Illustrations of mis-registrations:



Penalized Least Square Framework

- A natural solution to register trajectories is: Γ is the group of diffeomorphisms of $[0, 1]$ –

$$\hat{\gamma} = \operatorname*{arginf}_{\gamma \in \Gamma} \int_0^1 d_m(\alpha_1(t), \alpha_2(\gamma(t)))^2 dt$$

  Analogous to minimizing $\mathbb{L}^2$ norm for Euclidean curves.
- Prone to the pinching effect.
- Penalized Least Squares:

$$\hat{\gamma} = \operatorname*{arginf}_{\gamma \in \Gamma} \left(\int_0^1 d_m(\alpha_1(t), \alpha_2(\gamma(t)))^2 dt + \lambda \mathcal{R}(\gamma) \right) .$$

  Asymmetric solutions; difficulty in choosing λ; the quality of registration is bad.
- The main problem:

$$\int_0^1 d_m(\alpha_1(t), \alpha_2(t))\, dt \neq \int_0^1 d_m(\alpha_1(\gamma(t)), \alpha_2(\gamma(t)))\, dt$$

- Need a metric on the space of trajectories that is invariant to the action of the time warping group.

Elastic Registration Between Trajectories

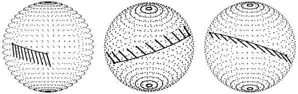
- Problem Statement: Given any two trajectories, say α_1 and α_2, we are interested in finding function γ such that the points $\alpha_2(\gamma_i(t))$ is matched optimally to $\alpha_1(t)$, for all t.
- What about SRVF? The standard SRVF is well defined for this situation also: for any $\alpha : [0,1] \to M$, define

$$q(t) = \frac{\dot{\alpha}(t)}{\sqrt{|\dot{\alpha}(t)|}} \in T_{\alpha(t)}(M) \ .$$

  However, this is a tangent vector field along α.
- We can't easily compare two SRVFs as they are two vector fields along two different curves. They lie in different tangent spaces.
- We need to bring them to the same coordinate system.

Parallel Transport of Tangent Vectors

- Parallel Transport: Take tangent vectors along given paths. Notation: $(v)_{p_1 \to p_2}$ – vector v is transported from p_1 to p_2 along a geodesic.



- Definition: Given a path α and a tangent vector $v_0 \in T_{\alpha(0)}(M)$, construct a vector field $v(t) \in T_{\alpha(t)}(M)$ such that: (1) $v(0) = v_0$, and (2) the covariant derivative of $v(t)$ is zero everywhere. Then, $v(1)$ is the parallel transport of v_0 along α to $\alpha(1)$.
- Parallel transport preserves inner product between any two vectors. Thus, it preserves the norm of a vector. That is,

$$\|v\| = \|(v)_{p_1 \to p_2}\| .$$

Approach: Transported SRVF

Different Choices:

- Global Transport: Transport all the SRVFs as tangent vectors to the same tangent space $T_c(M)$, using geodesic paths.
  The transported vectors form a curve in the space $T_c(M)$. Now, we are studying curves in a Hilbert space and standard techniques apply.
  The simplifies the problem but approximates the geometry.
- Local Transport: Transport all the SRVFs to the tangent space of the starting point of the curve $T_{\alpha(0)}(M)$, using geodesic paths.
  Each trajectory is represented by a curve in the tangent space $T_{\alpha(0)}(M)$. The set of such curves is called a vector bundle on M.
  This simplifies the geometry a little bit but mostly preserves the geometry.
- No Transport: Study them as curves in the tangent bundle of M – TM. No simplification. Full use of geometry.

Outline

Transported SRVF

- Definition 1: Transported Square-Root Vector Fields (TSRVF):

$$h_\alpha(t) = \frac{\dot{\alpha}(t)_{\alpha(t)\to c}}{\sqrt{|\dot{\alpha}(t)|}} \in T_c(M), \quad h_\alpha \in \mathbb{L}^2([0,1], T_c(M))$$

- The TSRVF of a re-parameterized trajectory $\alpha \circ \gamma$ is $h_{\alpha\circ\gamma} = (h_\alpha \circ \gamma)\sqrt{\dot{\gamma}} = (h_\alpha, \gamma)$. Commutative Diagram

$$\begin{array}{ccc} \alpha & \xrightarrow{\text{TSRVF}} & h_\alpha \\ \text{Group action by } \Gamma \downarrow & & \downarrow \text{Group action by } \Gamma \\ (\alpha \circ \gamma) & \xrightarrow[\text{TSRVF}]{} & (h_\alpha, \gamma) \end{array}$$

Transported SRVF: Properties

- If $M = \mathbb{R}^n$, then TSRVF is exactly the SRVF discussed earlier.
- Given $\alpha(0)$ (starting point) and a TSRVF h_α, we can reconstruct the trajectory α completely:

$$\alpha(t) = \oint_0^t h_\alpha(s)|h_\alpha(s)| \, ds$$

- The set of all TSRVF is $\mathbb{L}^2([0,1], T_c(M))$, a vector space.
- Distance between two trajectories is defined to be the $\mathbb{L}^2$ distance between their TSRVFs:

$$d_h(h_{\alpha_1}, h_{\alpha_2}) \equiv \left(\int_0^1 |h_{\alpha_1}(t) - h_{\alpha_2}(t)|^2 dt \right)^{\frac{1}{2}} .$$

- Lemma: For any $\alpha_1, \alpha_2 \in \mathcal{M}$ and $\gamma \in \Gamma$, the distance d_h satisfies

$$d_h(h_{\alpha_1 \circ \gamma}, h_{\alpha_2 \circ \gamma)}) = d_h(h_{\alpha_1}, h_{\alpha_2}).$$

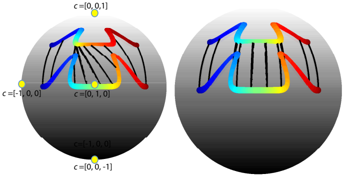
  In geometric terms, this implies that the action of Γ on the set of trajectories d_h is by isometries.

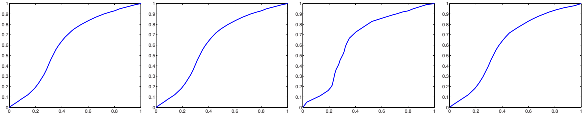
Pairwise Temporal Registration

- This sets up the pairwise temporal registration solution:

$$\gamma^* = \operatorname*{arginf}_{\gamma \in \Gamma} d_h(h_{\alpha_1}, h_{\alpha_2 \circ \gamma}) .$$

- Example 1: Spherical Trajectories
  $M = \mathbb{S}^2$.

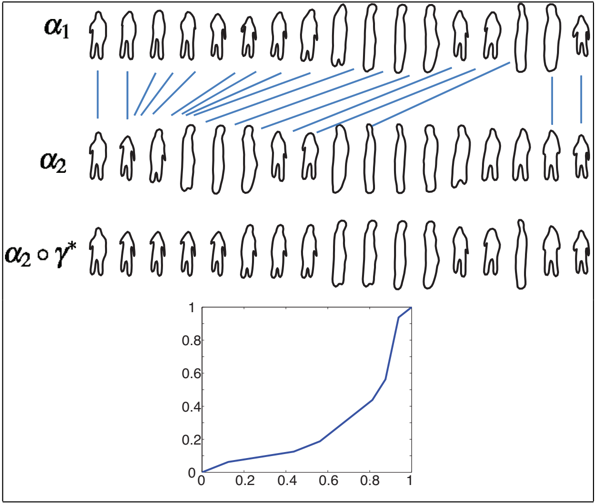




$c = [0, 0, 1]$ $c = [0, 0, -1]$ $c = [-1, 0, 0]$ $c = [0, 1, 0]$

Pairwise Temporal Registration

- Example 2: Shape trajectories
  M = Kendall's shape space of planar shapes.



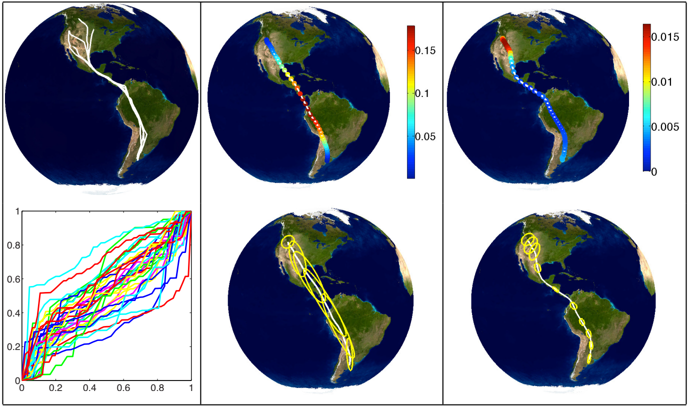
Registration of Multiple Trajectories

Karcher Mean of Multiple Trajectories:
Compute the Karcher Mean of $\{\alpha_i(0)\}s$ and set it to be $\mu(0)$.

1. Initialization step: Select μ to be one of the original trajectories and compute its TSRVF h_μ.
2. Align each h_{α_i}, $i = 1, ..., n$, to h_μ according to pairwise registration. That is, solve for γ_i^* using the DP algorithm and set $\tilde{\alpha}_i = \alpha_i \circ \gamma_i^*$.
3. Compute TSRVFs of the warped trajectories, $h_{\tilde{\alpha}_i}$, $i = 1, 2, \ldots, n$, and update h_μ as a curve in $T_c(M)$ according to: $h_\mu(t) = \frac{1}{n}\sum_{i=1}^n h_{\tilde{\alpha}_i}(t)$.
4. Define μ to be the integral curve associated with a time-varying vector field on M generated using h_μ, i.e. $\frac{d\mu(t)}{dt} = (h_\mu)(t)_{c\to\mu(t)}$, and the initial condition $\mu(0)$.
5. Compute $E = \sum_{i=1}^n d_s([h_\mu], [h_{\alpha_i}])^2 = \sum_{i=1}^n d_h(h_\mu, h_{\tilde{\alpha}_i})^2$ and check it for convergence. If not converged, return to step 2.

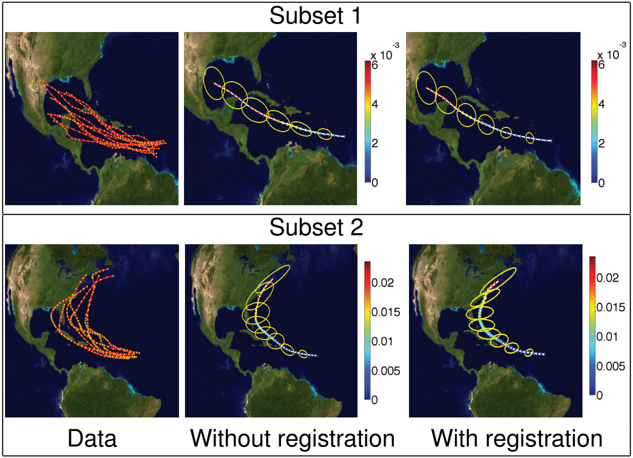
Registration: Examples

Bird Migration Data:



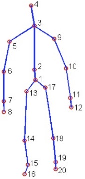
Registration: Examples

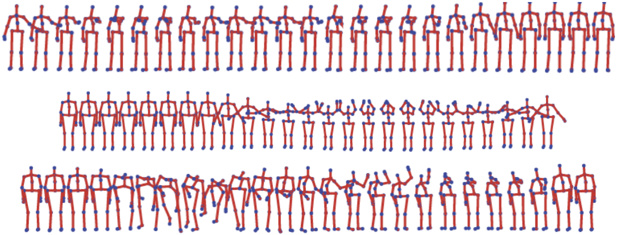
Hurricane Trajectory Data:



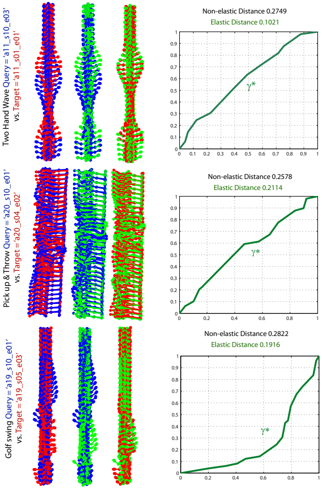
Shape Trajectories

Application: Activity recognition using depth sensing (Kinect)





Shape Trajectories



Action Classification

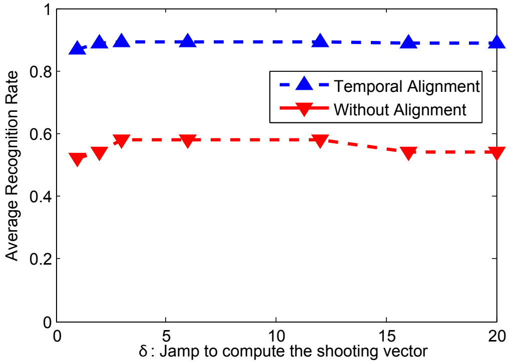
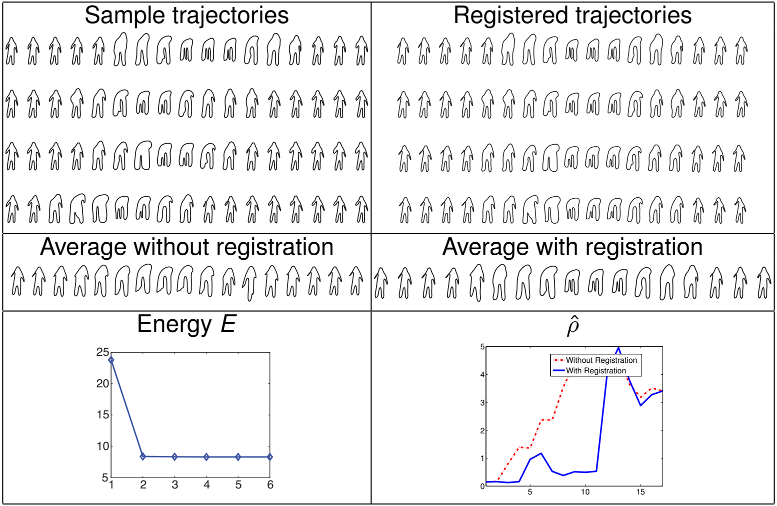


Figure: Impact of the temporal alignment and the changes in δ on SVM-based classification accuracy.

Summaries of Trajectories

Sample Mean: Shape Trajectories



Limitations of this TSRVF

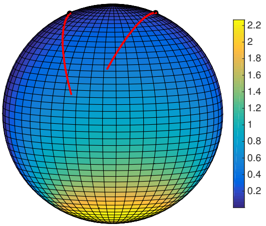


Figure shows the variability in distance between trajectories as the reference point changes over $\mathbb{S}^2$. The color at each point denotes the distance with that point as reference.

- One needs to choose a reference point c, and the results may depend on this choice.
- The parallel transport to c can distort tangents, especially if the data is distributed over the whole manifold.

Outline

New TSRVF

- Definition 2: TSRVF
  - For each trajectory choose its starting point as the reference.
  - Transport scaled velocity vectors along the trajectories to their starting points:

$$h_\alpha(t) = \frac{\dot{\alpha}(t)_{\alpha(t)\to\alpha(0)}}{\sqrt{|\dot{\alpha}(t)|}} \in T_c(M), \quad h_\alpha \in \mathbb{L}^2([0,1], T_{\alpha(0)}(M))$$

  Each trajectory is represented by a starting point $\alpha(0)$ and a TSRVF h_α at $\alpha(0)$.

- The set of all such representations is a *vector bundle* over M. At each point, we have an $\mathbb{L}^2$ space.
  Vector bundle: $\mathbb{B} = \coprod_{p\in M} \mathbb{B}_p = \coprod_{p\in M} \mathbb{L}^2([0,1], T_p(M))$.
- For an element $(p, q(\cdot))$ in $\mathbb{B}$, where $p \in M$, $q \in \mathbb{B}_p$, we naturally identify the tangent space at (p, q) to be: $T_{(p,q)}(\mathbb{B}) \cong T_p(M) \oplus \mathbb{B}_p$.
- Invariant Riemannian Metric:

$$\langle (u_1, w_1(\cdot)), (u_2, w_2(\cdot)) \rangle = (u_1 \cdot u_2) + \int_0^1 (w_1(\tau) \cdot w_2(\tau))\, d\tau, \quad (1)$$

Geodesics Under this Metric

Theorem

A parameterized path $[0,1] \to \mathbb{B}$ *given by* $s \mapsto (p(s), q(s,\tau))$ *on* $\mathbb{B}$ *(where the variable* τ *corresponds to the parametrization in* $\mathbb{B}_p$*), is a geodesic in* $\mathbb{B}$ *if and only if:*

$$\begin{aligned} \nabla_{p_s} p_s + \int_0^1 R(q, \nabla_{p_s} q)(p_s) d\tau &= 0 \quad \text{for every } s, \\ \nabla_{p_s}(\nabla_{p_s} q)(s,\tau) &= 0 \quad \text{for every } s, \tau. \end{aligned} \tag{2}$$

Here $R(\cdot,\cdot)(\cdot)$ *denotes the Riemannian curvature tensor,* p_s *denotes* dp/ds*, and* ∇_{p_s} *denotes the covariant differentiation of tangent vectors on tangent space* $T_{p(s)}(M)$.

Exponential Map

Let the initial point be $(p(0), q(0)) \in \mathbb{B}$ and the tangent vector be $(u, w) \in T_{(p(0),q(0))}(\mathbb{B})$. We have $p_s(0) = u$, $\nabla_{p_s} q(s)|_{s=0} = w$. We will approximate this map using n steps and let $\epsilon = \frac{1}{n}$. Then, for $i = 1, \cdots, n$ the exponential map $(p(i\epsilon), q(i\epsilon)) = \exp_{(p(0),q(0))}(i\epsilon(u, w))$ is given as:

1. Set $p(\epsilon) = \exp_{p(0)}(\epsilon p_s(0))$, where $p_s(0) = u$, and $q(\epsilon) = (q^{\|} + \epsilon w^{\|})$, where $q^{\|}$ and $w^{\|}$ are parallel transports of $q(0)$ and w along path p from $p(0)$ to $p(\epsilon)$, respectively.
2. For each i = 1,2,...,n-1, calculate

$$p_s(i\epsilon) = [p_s((i-1)\epsilon) + \epsilon \nabla_{p_s} p_s((i-1)\epsilon)]_{p((i-1)\epsilon) \to p(i\epsilon)},$$

   where $\nabla_{p_s} p_s((i-1)\epsilon) = -R\left(q((i-1)\epsilon), \nabla_{p_s} q((i-1)\epsilon)\right)(p_s((i-1)\epsilon))$ is given by the first equation in Theorem 1. It is easy to show that $R\left(q((i-1)\epsilon), \nabla_{p_s} q((i-1)\epsilon)\right) = R\left(q^{\|} + \epsilon(i-1)w^{\|}, w^{\|}\right) = R\left(q^{\|}, w^{\|}\right)$, where $q^{\|} = q(0)_{p(0)\to p((i-1)\epsilon)}$, and $w^{\|} = w_{p(0)\to p((i-1)\epsilon)}$.
3. Obtain $p((i+1)\epsilon) = \exp_{p(i\epsilon)}(\epsilon p_s(i\epsilon))$, and $q((i+1)\epsilon) = q^{\|} + (i+1)\epsilon w^{\|}$, where $q^{\|} = q(0)_{p(0)\to p((i+1)\epsilon)}$, and $w^{\|} = w_{p(0)\to p((i+1)\epsilon)}$.

Shooting Algorithm for Computing Geodesics

Given $(p_1, q_2), (p_2, q_2) \in \mathbb{B}$, select one point, say (p_1, q_1), as the starting point and the other, (p_2, q_2), as the target point. The shooting algorithm for calculating the geodesic from (p_1, q_1) to (p_2, q_2) is:

1. Initialize the shooting direction: find the tangent vector u at p_1 such that the exponential map $\exp_{p_1}(u) = p_2$ on the manifold M. Parallel transport q_2 to the tangent space of p_1 along the shortest geodesic between p_1 and p_2, denoted as $q_2^{\parallel}$. Initialize $w = q_2^{\parallel} - q_1$. Now we have a pair $(u, w) \in T_{(p_1, q_1)}(\mathbb{B})$.
2. Construct a geodesic starting from (p_1, q_1) in the direction (u, w) using the numerical exponential map in previous page. Let us denote this geodesic path as $(x(s), v(s))$, where s is the time parameter for the geodesic path.
3. If $(x(1), v(1)) = (p_2, q_2)$, we are done. If not, measure the discrepancy between $(x(1), v(1))$ and (p_2, q_2) using a simple measure, e.g. the $\mathbb{L}^2$ distance.
4. Iteratively, update the shooting direction (u, w) to reduce the discrepancy to zero. This update can be done using a two-stage approach: (1) fix u and update w until converge; (2) fix w and update u until converge.

- The length of a geodesic path is given by:

$$d((p_1, q_1), (p_2, q_2)) = \sqrt{l_x^2 + \int_0^1 |q_{1,x}^{\parallel}(t) - q_2(t)|^2 dt} \ .$$

- For any two trajectories $\alpha_1, \alpha_2 \in \mathcal{F}$, and the corresponding representation $(p_1, q_{\alpha_1}), (p_2, q_{\alpha_2}) \in \mathbb{B}$, the metric d satisfies

$$d((p_1, q_{\alpha_1 \circ \gamma}), (p_2, q_{\alpha_2 \circ \gamma})) = d((p_1, q_{\alpha_1}), (p_2, q_{\alpha_2})) \ ,$$

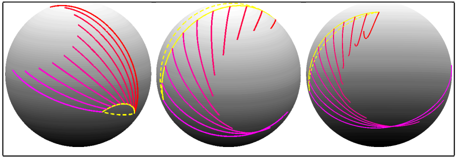
  for any $\gamma \in \Gamma$.
- Registration problem:

$$\hat{\gamma} = \inf_{\gamma \in \Gamma} d((p_1, q_1), (p_2, (q_2 \circ \gamma)\sqrt{\dot{\gamma}}))$$

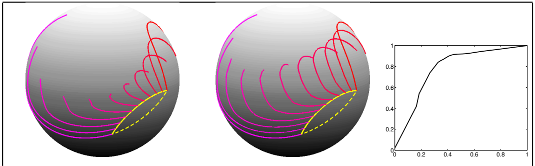
Examples: Spherical Trajectories

If $M = \mathbb{S}^k$, then the computations can be simplified. We know that the base path x is a circle (not necessarily a great circle) and therefor one can search for that directly. Given a base path, the evolution of TSRVF along that path is straightforward.

- Examples of geodesic paths:



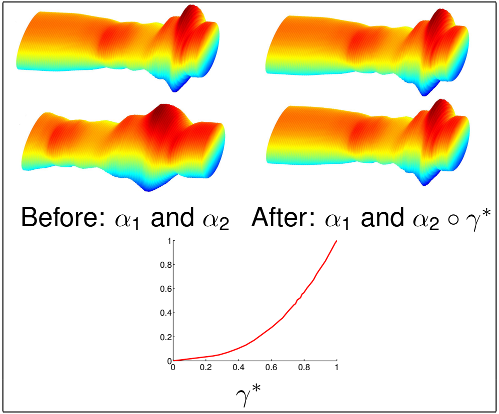
- Example of registration:



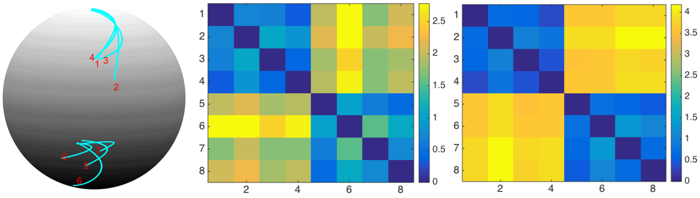
Before registration After registration γ^*

Registration Example: Covariance Trajectories

M = SPDM(3). Each SPDM can be visualized as an ellipse.



Comparison of Old and New TSRVF



(a) (b) (c)

Figure: Metric comparisons: (a) shows eight simulated spherical trajectories, (b) shows the pairwise distance matrix calculated using older TSRVF and (c) shows the distance matrix calculated using new TSRVF. The trajectories are labeled (1-8), with corresponding columns and rows in distance matrices.

Real Data Examples

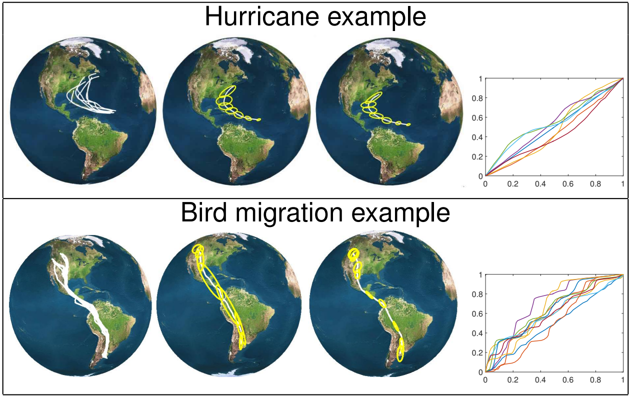


Figure: Comparison of the cross-sectional mean (column 2) and the amplitude mean (column 3) for hurricane and bird migration data (left panel). Yellow ellipsoids in column 2 and 3 represent the cross-sectional variance along the mean trajectory. The last column shows the estimated phases $\{\gamma_i^*\}$.

FPCA on Manifolds

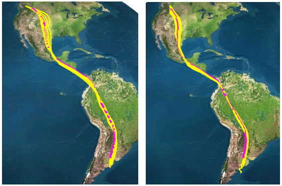
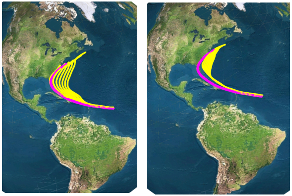
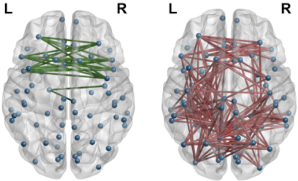
First two PCs for bird data (41.90%)	First two PCs for hurricane data (69.43%)
	

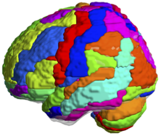
Figure: PCA results for bird migration (left panel) and hurricane data (right panel). The number in the parenthesis shows the percentage of variation explained by the first two PCs.

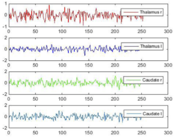
Dynamical Functional Connectivity in Human Brain

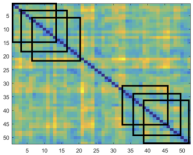
- Functional Connectivity: Statistical dependencies in signals generated by distant regions of brain under certain neurophysiological events, as measured by fMRI data.



- Generating covariance trajectories:

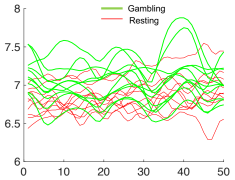




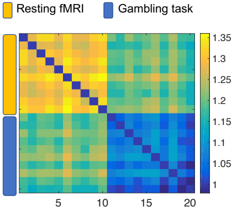


Registration Example: Covariance Trajectories

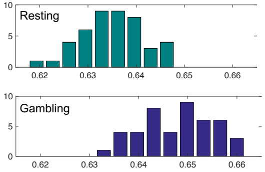
Brain functional connectivity using covariance trajectories



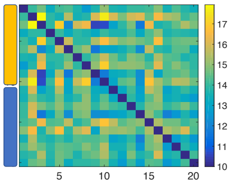
(a) Determinant part



(b) Pairwise elastic distances d_s



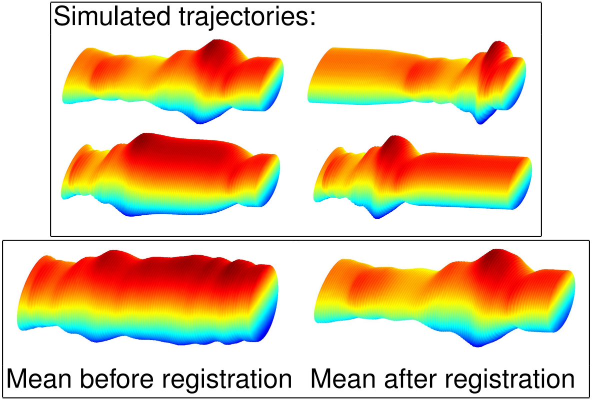
(c) Hist. of $(d_h - d_s)/\max(d_h, d_s)$



(d) Pairwise distances based on $\log_E$

Summaries of Trajectories

Sample Mean: Covariance Trajectories



Summary

- The shape analysis of trajectories on manifold can be handled using SRVFs but requires parallel transport.
- The Transported SRVFs can be used for registering, averaging, and analyzing (PCA) trajectories.
- Temporal alignment is important in applications to result in rate-invariant classification.