SHAPE ANALYSIS OF TRAJECTORIES ON MANIFOLDS

Anuj Srivastava

Department of Statistics, Florida State University
1 Introduction and Motivation

2 Current Ideas

3 Elastic Framework
   - Approach 1: Global Transport
   - Approach II: Local Transport

4 Conclusion
Interested in curves of the type $\alpha : [0, 1] \to M$, where $M$ is a nonlinear Riemannian manifold. Longitudinal data on manifolds.

- **Spherical Trajectories**: $M = S^d$ a unit sphere
  Directional data, geographical data.

- **Covariance Trajectories**: $M = \mathbb{P}$, the set of symmetric, positive definite matrices.
  Brain connectivity data

- **Shape Trajectories**: $M = S$ a shape space
  Video data, action recognition.

- **Graph Trajectories**: $M = \mathcal{G}$ a space of graphs
  Social networks, recommender systems.
Some of the Applications

- Activity recognition using video and depth sensing.
- Hurricane trajectories.
- Bird migration data.
- Dynamical functional connectivity analysis
- Biological growth data
Consider an arbitrary Riemannian manifold $M$. Let $d_m$ be the geodesic distance on $M$. In order to compare any two trajectories $\alpha_1, \alpha_2 : [0, 1] \rightarrow M$, one use the metric:

$$d_x(\alpha_1, \alpha_2) = \int_0^1 d_m(\alpha_1(t), \alpha_2(t)) \, dt.$$  

However, the given data may lack temporal registration. We need to register the trajectories.

Illustrations of mis-registrations:
A natural solution to register trajectories is: $\Gamma$ is the group of diffeomorphisms of $[0, 1]$ –

$$\hat{\gamma} = \arg\inf_{\gamma \in \Gamma} \int_0^1 d_m(\alpha_1(t), \alpha_2(\gamma(t)))^2 \, dt$$

Analogous to minimizing $L^2$ norm for Euclidean curves.

- Prone to the pinching effect.
- Penalized Least Squares:

$$\hat{\gamma} = \arg\inf_{\gamma \in \Gamma} \left( \int_0^1 d_m(\alpha_1(t), \alpha_2(\gamma(t)))^2 \, dt + \lambda R(\gamma) \right).$$

Asymmetric solutions; difficulty in choosing $\lambda$; the quality of registration is bad.

- The main problem:

$$\int_0^1 d_m(\alpha_1(t), \alpha_2(t)) \, dt \neq \int_0^1 d_m(\alpha_1(\gamma(t)), \alpha_2(\gamma(t))) \, dt$$

- Need a metric on the space of trajectories that is invariant to the action of the time warping group.
Problem Statement: Given any two trajectories, say $\alpha_1$ and $\alpha_2$, we are interested in finding function $\gamma$ such that the points $\alpha_2(\gamma_i(t))$ is matched optimally to $\alpha_1(t)$, for all $t$.

What about SRVF? The standard SRVF is well defined for this situation also: for any $\alpha : [0, 1] \rightarrow M$, define

$$q(t) = \frac{\dot{\alpha}(t)}{\sqrt{|\dot{\alpha}(t)|}} \in T_{\alpha(t)}(M).$$

However, this is a tangent vector field along $\alpha$.

We can’t easily compare two SRVFs as they are two vector fields along two different curves. They lie in different tangent spaces.

We need to bring them to the same coordinate system.
Parallel Transport of Tangent Vectors

- **Parallel Transport**: Take tangent vectors along given paths. Notation: \((v)_{p_1 \rightarrow p_2}\) – vector \(v\) is transported from \(p_1\) to \(p_2\) along a geodesic.

- **Definition**: Given a path \(\alpha\) and a tangent vector \(v_0 \in T_{\alpha(0)}(M)\), construct a vector field \(v(t) \in T_{\alpha(t)}(M)\) such that: (1) \(v(0) = v_0\), and (2) the covariant derivative of \(v(t)\) is zero everywhere. Then, \(v(1)\) is the parallel transport of \(v_0\) along \(\alpha\) to \(\alpha(1)\).

- Parallel transport preserves inner product between any two vectors. Thus, it preserves the norm of a vector. That is,

\[
\|v\| = \|(v)_{p_1 \rightarrow p_2}\|.
\]
Approach: Transported SRVF

Different Choices:

- **Global Transport**: Transport all the SRVFs as tangent vectors to the same tangent space $T_c(M)$, using geodesic paths. The transported vectors form a curve in the space $T_c(M)$. Now, we are studying curves in a Hilbert space and standard techniques apply. The simplifies the problem but approximates the geometry.

- **Local Transport**: Transport all the SRVFs to the tangent space of the starting point of the curve $T_{\alpha(0)}(M)$, using geodesic paths. Each trajectory is represented by a curve in the tangent space $T_{\alpha(0)}(M)$. The set of such curves is called a vector bundle on $M$. This simplifies the geometry a little bit but mostly preserves the geometry.

- **No Transport**: Study them as curves in the tangent bundle of $M – TM$. No simplification. Full use of geometry.
Outline

1. Introduction and Motivation
2. Current Ideas
3. Elastic Framework
   - Approach 1: Global Transport
   - Approach II: Local Transport
4. Conclusion
Definition 1: Transported Square-Root Vector Fields (TSRVF):

\[ h_\alpha(t) = \frac{\dot{\alpha}(t)}{\sqrt{|\dot{\alpha}(t)|}} \in T_c(M), \quad h_\alpha \in L^2([0, 1], T_c(M)) \]

The TSRVF of a re-parameterized trajectory \( \alpha \circ \gamma \) is

\[ h_{\alpha \circ \gamma} = (h_\alpha \circ \gamma) \sqrt{\gamma} = (h_\alpha, \gamma). \]

Commutative Diagram

\[
\begin{array}{ccc}
\alpha & \xrightarrow{\text{TSRVF}} & h_\alpha \\
\downarrow & & \downarrow \\
(\alpha \circ \gamma) & \xrightarrow{\text{TSRVF}} & (h_\alpha, \gamma)
\end{array}
\]
If $M = \mathbb{R}^n$, then TSRVF is exactly the SRVF discussed earlier.

Given $\alpha(0)$ (starting point) and a TSRVF $h_\alpha$, we can reconstruct the trajectory $\alpha$ completely:

$$\alpha(t) = \int_0^t h_\alpha(s)|h_\alpha(s)| \, ds$$

The set of all TSRVF is $L^2([0, 1], T_c(M))$, a vector space.

Distance between two trajectories is defined to be the $L^2$ distance between their TSRVFs:

$$d_h(h_{\alpha_1}, h_{\alpha_2}) \equiv \left( \int_0^1 |h_{\alpha_1}(t) - h_{\alpha_2}(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

**Lemma:** For any $\alpha_1, \alpha_2 \in \mathcal{M}$ and $\gamma \in \Gamma$, the distance $d_h$ satisfies

$$d_h(h_{\alpha_1 \circ \gamma}, h_{\alpha_2 \circ \gamma}) = d_h(h_{\alpha_1}, h_{\alpha_2}).$$

In geometric terms, this implies that the action of $\Gamma$ on the set of trajectories $d_h$ is by isometries.
This sets up the pairwise temporal registration solution:

$$\gamma^* = \arg\inf_{\gamma \in \Gamma} d_h(h_{\alpha_1}, h_{\alpha_2} \circ \gamma).$$

**Example 1:** Spherical Trajectories

$$M = S^2.$$
Example 2: Shape trajectories

$M = \text{Kendall's shape space of planar shapes.}$
Karcher Mean of Multiple Trajectories:
Compute the Karcher Mean of \{\alpha_i(0)\}s and set it to be \mu(0).

1. **Initialization step**: Select \mu to be one of the original trajectories and compute its TSRVF \( h_\mu \).

2. Align each \( h_{\alpha_i}, \, i = 1, \ldots, n \), to \( h_\mu \) according to pairwise registration. That is, solve for \( \gamma_i^* \) using the DP algorithm and set \( \tilde{\alpha}_i = \alpha_i \circ \gamma_i^* \).

3. Compute TSRVFs of the warped trajectories, \( h_{\tilde{\alpha}_i}, \, i = 1, 2, \ldots, n \), and update \( h_\mu \) as a curve in \( T_c(M) \) according to:
\[
h_\mu(t) = \frac{1}{n} \sum_{i=1}^{n} h_{\tilde{\alpha}_i}(t).
\]

4. Define \( \mu \) to be the integral curve associated with a time-varying vector field on \( M \) generated using \( h_\mu \), i.e. \( \frac{d\mu(t)}{dt} = (h_\mu)(t)_{c\rightarrow \mu(t)} \), and the initial condition \( \mu(0) \).

5. Compute \( E = \sum_{i=1}^{n} d_s([h_\mu], [h_{\alpha_i}])^2 = \sum_{i=1}^{n} d_h(h_\mu, h_{\tilde{\alpha}_i})^2 \) and check it for convergence. If not converged, return to step 2.
Bird Migration Data:
Registration: Examples

Hurricane Trajectory Data:

<table>
<thead>
<tr>
<th>Subset 1</th>
<th>Data</th>
<th>Without registration</th>
<th>With registration</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Subset 1 Data" /></td>
<td><img src="image2.png" alt="Subset 1 Without registration" /></td>
<td><img src="image3.png" alt="Subset 1 With registration" /></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Subset 2</th>
<th>Data</th>
<th>Without registration</th>
<th>With registration</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image4.png" alt="Subset 2 Data" /></td>
<td><img src="image5.png" alt="Subset 2 Without registration" /></td>
<td><img src="image6.png" alt="Subset 2 With registration" /></td>
<td></td>
</tr>
</tbody>
</table>
Shape Trajectories

Application: Activity recognition using depth sensing (Kinect)
Shape Trajectories

Two Hand Wave Query = 'a11_s10_e03' vs. Target = 'a11_s01_e01

Pick up & Throw Query = 'a20_s10_e01' vs. Target = 'a20_s04_e02'

Golf swing Query = 'a19_s10_e01' vs. Target = 'a19_s05_e03'
Figure: Impact of the temporal alignment and the changes in $\delta$ on SVM-based classification accuracy.
Summaries of Trajectories

Sample Mean: Shape Trajectories

<table>
<thead>
<tr>
<th>Sample trajectories</th>
<th>Registered trajectories</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Sample trajectories without registration" /></td>
<td><img src="image2" alt="Registered trajectories" /></td>
</tr>
<tr>
<td><img src="image3" alt="Average without registration" /></td>
<td><img src="image4" alt="Average with registration" /></td>
</tr>
</tbody>
</table>

Energy $E$

![Energy plot](image5)

$\hat{\rho}$

![Energy density plot](image6)
Figure shows the variability in distance between trajectories as the reference point changes over $S^2$. The color at each point denotes the distance with that point as reference.

- One needs to choose a reference point $c$, and the results may depend on this choice.
- The parallel transport to $c$ can distort tangents, especially if the data is distributed over the whole manifold.
Outline

1. Introduction and Motivation
2. Current Ideas
3. Elastic Framework
   - Approach 1: Global Transport
   - Approach II: Local Transport
4. Conclusion
New TSRVF

- Definition 2: TSRVF
  - For each trajectory choose its starting point as the reference.
  - Transport scaled velocity vectors along the trajectories to their starting points:

\[
h_\alpha(t) = \frac{\dot{\alpha}(t)_{\alpha(t)\rightarrow\alpha(0)}}{\sqrt{|\dot{\alpha}(t)|}} \in T_c(M), \quad h_\alpha \in L^2([0, 1], T_{\alpha(0)}(M))
\]

Each trajectory is represented by a starting point \( \alpha(0) \) and a TSRVF \( h_\alpha \) at \( \alpha(0) \).

- The set of all such representations is a vector bundle over \( M \). At each point, we have an \( L^2 \) space.
  - Vector bundle: \( B = \bigsqcup_{p \in M} B_p = \bigsqcup_{p \in M} L^2([0, 1], T_p(M)) \).

- For an element \( (p, q(\cdot)) \) in \( B \), where \( p \in M, q \in B_p \), we naturally identify the tangent space at \( (p, q) \) to be: \( T_{(p, q)}(B) \cong T_p(M) \oplus B_p \).

- Invariant Riemannian Metric:

\[
\langle (u_1, w_1(\cdot)), (u_2, w_2(\cdot)) \rangle = (u_1 \cdot u_2) + \int_0^1 (w_1(\tau) \cdot w_2(\tau)) \, d\tau,
\]

(1)
A parameterized path $[0, 1] \to \mathbb{B}$ given by $s \mapsto (p(s), q(s, \tau))$ on $\mathbb{B}$ (where the variable $\tau$ corresponds to the parametrization in $\mathbb{B}_p$), is a geodesic in $\mathbb{B}$ if and only if:

$$\nabla_{p_s} p_s + \int_0^1 R(q, \nabla_{p_s} q)(p_s) d\tau = 0 \quad \text{for every } s,$$

$$\nabla_{p_s} (\nabla_{p_s} q)(s, \tau) = 0 \quad \text{for every } s, \tau.$$  \hfill (2)

Here $R(\cdot, \cdot)(\cdot)$ denotes the Riemannian curvature tensor, $p_s$ denotes $dp/ds$, and $\nabla_{p_s}$ denotes the covariant differentiation of tangent vectors on tangent space $T_{p(s)}(M)$. 

**Theorem**
Let the initial point be \((p(0), q(0)) \in \mathbb{B}\) and the tangent vector be \((u, w) \in T_{(p(0), q(0))}(\mathbb{B})\). We have \(p_s(0) = u, \nabla_{p_s} q(s)|_{s=0} = w\). We will approximate this map using \(n\) steps and let \(\epsilon = \frac{1}{n}\). Then, for \(i = 1, \cdots, n\) the exponential map \((p(i\epsilon), q(i\epsilon)) = \exp_{(p(0), q(0))} (i\epsilon (u, w))\) is given as:

1. Set \(p(\epsilon) = \exp_{p(0)} (\epsilon p_s(0))\), where \(p_s(0) = u\), and \(q(\epsilon) = (q^\parallel + \epsilon w^\parallel)\), where \(q^\parallel\) and \(w^\parallel\) are parallel transports of \(q(0)\) and \(w\) along path \(p\) from \(p(0)\) to \(p(\epsilon)\), respectively.

2. For each \(i = 1, 2, \ldots, n-1\), calculate

\[
p_s(i\epsilon) = [p_s((i - 1)\epsilon) + \epsilon \nabla_{p_s} p_s((i - 1)\epsilon)]_{p((i-1)\epsilon)\rightarrow p(i\epsilon)},
\]

where \(\nabla_{p_s} p_s((i - 1)\epsilon) = -R(q((i - 1)\epsilon), \nabla_{p_s} q((i - 1)\epsilon)) (p_s((i - 1)\epsilon))\) is given by the first equation in Theorem 1. It is easy to show that

\[
R(q((i - 1)\epsilon), \nabla_{p_s} q((i - 1)\epsilon)) = R\left(q^\parallel + \epsilon (i - 1)w^\parallel, w^\parallel\right) = R\left(q^\parallel, w^\parallel\right),
\]

where \(q^\parallel = q(0)_{p(0)\rightarrow p((i-1)\epsilon)}\), and \(w^\parallel = w_{p(0)\rightarrow p((i-1)\epsilon)}\).

3. Obtain \(p((i + 1)\epsilon) = \exp_{p(i\epsilon)} (\epsilon p_s(i\epsilon))\), and \(q((i + 1)\epsilon) = q^\parallel + (i + 1)\epsilon w^\parallel\), where \(q^\parallel = q(0)_{p(0)\rightarrow p((i+1)\epsilon)}\), and \(w^\parallel = w_{p(0)\rightarrow p((i+1)\epsilon)}\).
Given \((p_1, q_2), (p_2, q_2) \in \mathbb{B}\), select one point, say \((p_1, q_1)\), as the starting point and the other, \((p_2, q_2)\), as the target point. The shooting algorithm for calculating the geodesic from \((p_1, q_1)\) to \((p_2, q_2)\) is:

1. **Initialize the shooting direction:** find the tangent vector \(u\) at \(p_1\) such that the exponential map \(\exp_{p_1}(u) = p_2\) on the manifold \(M\). Parallel transport \(q_2\) to the tangent space of \(p_1\) along the shortest geodesic between \(p_1\) and \(p_2\), denoted as \(q_2^\parallel\). Initialize \(w = q_2^\parallel - q_1\). Now we have a pair \((u, w) \in T_{(p_1, q_1)}(\mathbb{B})\).

2. **Construct a geodesic starting from** \((p_1, q_1)\) in the direction \((u, w)\) using the numerical exponential map in previous page. Let us denote this geodesic path as \((x(s), v(s))\), where \(s\) is the time parameter for the geodesic path.

3. **If** \((x(1), v(1)) = (p_2, q_2)\), we are done. If not, measure the discrepancy between \((x(1), v(1))\) and \((p_2, q_2)\) using a simple measure, e.g. the \(L^2\) distance.

4. **Iteratively,** update the shooting direction \((u, w)\) to reduce the discrepancy to zero. This update can be done using a two-stage approach: (1) fix \(u\) and update \(w\) until converge; (2) fix \(w\) and update \(u\) until converge.
The length of a geodesic path is given by:

\[ d((p_1, q_1), (p_2, q_2)) = \sqrt{l_x^2 + \int_0^1 |q_{1,x}(t) - q_2(t)|^2 dt} \]

For any two trajectories \( \alpha_1, \alpha_2 \in \mathcal{F} \), and the corresponding representation \((p_1, q_{\alpha_1}), (p_2, q_{\alpha_2}) \in \mathcal{B}\), the metric \( d \) satisfies

\[ d((p_1, q_{\alpha_1 \circ \gamma}), (p_2, q_{\alpha_2 \circ \gamma})) = d((p_1, q_{\alpha_1}), (p_2, q_{\alpha_2})) \]

for any \( \gamma \in \Gamma \).

Registration problem:

\[ \hat{\gamma} = \inf_{\gamma \in \Gamma} d((p_1, q_1), (p_2, (q_2 \circ \gamma) \sqrt{\hat{\gamma}})) \]
Examples: Spherical Trajectories

If $M = S^k$, then the computations can be simplified. We know that the base path $x$ is a circle (not necessarily a great circle) and therefore one can search for that directly. Given a base path, the evolution of TSRVF along that path is straightforward.

- Examples of geodesic paths:

- Example of registration:

Before registration  After registration
$M = \text{SPDM}(3)$. Each SPDM can be visualized as an ellipse.
Figure: Metric comparisons: (a) shows eight simulated spherical trajectories, (b) shows the pairwise distance matrix calculated using older TSRVF and (c) shows the distance matrix calculated using new TSRVF. The trajectories are labeled (1-8), with corresponding columns and rows in distance matrices.
Real Data Examples

Figure: Comparison of the cross-sectional mean (column 2) and the amplitude mean (column 3) for hurricane and bird migration data (left panel). Yellow ellipsoids in column 2 and 3 represent the cross-sectional variance along the mean trajectory. The last column shows the estimated phases $\{\gamma_i^*\}$. 
### FPCA on Manifolds

<table>
<thead>
<tr>
<th>First two PCs for bird data (41.90%)</th>
<th>First two PCs for hurricane data (69.43%)</th>
</tr>
</thead>
</table>

**Figure:** PCA results for bird migration (left panel) and hurricane data (right panel). The number in the parenthesis shows the percentage of variation explained by the first two PCs.
Functional Connectivity: Statistical dependencies in signals generated by distant regions of brain under certain neurophysiological events, as measured by fMRI data.

Generating covariance trajectories:
Brain functional connectivity using covariance trajectories

(a) Determinant part

(b) Pairwise elastic distances $d_s$

(c) Hist. of $(d_h - d_s)/\max(d_h, d_s)$

(d) Pairwise distances based on $\log E$
Summaries of Trajectories

Sample Mean: Covariance Trajectories

Simulated trajectories:

Mean before registration  Mean after registration
The shape analysis of trajectories on manifold can be handled using SRVFs but requires parallel transport.

The Transported SRVFs can be used for registering, averaging, and analyzing (PCA) trajectories.

Temporal alignment is important in applications to result in rate-invariant classification.