Generalizing the Square Root Velocity Framework to Curves in a Riemannian Manifold Part II

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If we wish to analyze the "shapes" of a set of curves in $\mathbb{R}^n$, the family of elastic metrics and, in particular, the Square Root Velocity Function (SRVF), provides a useful framework, which Anuj has described earlier this week.

However, often the data we wish to analyze consists of curves in some other Riemannian manifold, which may not be isometric to $\mathbb{R}^n$. Examples include:

- A collection of hurricane paths on the surface of the earth.
- A collection of paths in a space of correlation matrices (i.e. positive definite symmetric matrices – PDSM) which may arise, for example, in the study of connectivity between different regions of the brain.
- A collection of paths in a shape space – which may arise in the analysis of videos.
- A collection of paths of probability distributions which, in certain cases, may be encoded as paths in the hyperbolic plane.

Main Goal

Given a Riemannian manifold M, we wish to put a Riemannian metric on the space of paths in M. This metric should have the following two important properties:

- It should be invariant under reparametrizations.
- It should be invariant under isometries (rigid motions) of M.

If $M = \mathbb{R}^n$ with the standard Euclidean metric, the family of elastic metrics satisfies these properties and, for the SRVF in particular, it is quite efficient to find geodesics in the space of paths, and make other statistical analyses as well.

How can we generalize the SRVF to paths in a more general Riemannian manifold M?

In the first part of this afternoon's lecture, Anuj has described two ways of extending the SRVF to analyze trajectories on a more general Reimannian manifold M:

- The "Transported Square Root Velocity Function", or TSRVF, in which each velocity vector is parallel translated along a minimal geodesic to a single chosen reference point in M.
- An approach in which each velocity vector to a path is parallel translated *along that path* to the initial point of that path. Since the representation space of all curves turns out to be a Hilbert space bundle over M, we will refer to this method as the "Hilbert Bundle Method".

In the next two slides we recap some of the advantages and disadvantages of these two methods. We then explore several other possible generalizations of the SRVF to trajectories in a Riemannian manifold.

Summing up the TSRVF

- The TSRVF is a straightforward generalization of the SRVF.
- Geodesics between parametrized curves are easy and efficient to compute.
- Registration is just as easy as the SRVF (dynamic programming).
- The TSRVF is not always well-defined, if the reference point p has a non-empty cut locus in M.
- The TSRVF introduces distortion for curves that stray far from the reference point.
- The TSRVF metric is not invariant under the isometries of M.

Summing up the Hilbert Bundle Approach

- This approach is a generalization of the SRVF.
- Geodesics between parametrized curves are harder to compute than the TSRVF, but still pretty fast.
- Registration is nearly as easy as the SRVF (dynamic programming).
- The Hilbert Bundle bijection is always well-defined.
- There is no arbitrary choice of reference point, hence no distortion for curves that stray far from a reference point.
- The Hilbert Bundle metric is invariant under isometries of M.

In 2015-2017, Celledoni et al. produced some nice work that generalized the SRVF to trajectories in Lie Groups[1] and Homogeneous Spaces[2].

- Their method avoids the arbitrariness and distortion resulting from the choice of a reference point;
- Their method uses left translation in a Lie group to compare vectors in different tangent spaces, which is computationally more efficient than using parallel translation along a path;
- For homogeneous spaces, their method was only implemented for sets of paths that all start at the same point.

[1] Celledoni, Eslitzbichler, Schmeding. Shape analysis on Lie groups with applications in computer animation, *J. Geom. Mech.*, 2015.

[2] Celledoni, Eidnes, Schmeding, Shape Analysis on Homogeneous Spaces, *arXiv*, 2017.

Start with Lie Groups

We will begin by describing the work of Celledoni et al. on Lie groups.

We will then proceed to some recent work of Su et al. on homogeneous spaces, which follows similar ideas to Celledoni et al., but improves it by making it possible to compare sets of paths which start at various different points in the homogeneous space M.

In the following slides we will repeatedly use the following notation. If M is any differentiable manifold, let $AC(I, M)$ denote the set of absolutely continuous functions $I \to M$.

Let G be a finite dimensional Lie group. Recall that a Lie group has two structures: it is a smooth manifold and also a group. Furthermore, these structures are compatible in the sense that the maps

$$G \to G \text{ defined by } g \mapsto g^{-1}$$

and

$$G \times G \to G \text{ defined by } (g, h) \mapsto gh$$

are both smooth. We define the Lie algebra of G by

$$\mathfrak{g} = T_1 G.$$

Given any element $g \in G$, define left translation by:

$$L_g : G \to G \text{ is defined by } L_g(h) = gh.$$

Left-Invariant Metric on G

We endow G with a left invariant Riemannian metric, i.e., a Riemannian metric with the property that for all $v, w \in T_h G$,

$$\langle dL_g v, dL_g w \rangle_{gh} = \langle v, w \rangle_h.$$

This is easily done by defining an arbitrary inner product on $T_1 G$, and then transporting it to all other tangent spaces of G using left translation.

SRVF on a Lie Group G

Definition

Given $\alpha \in AC(I, G)$, define $q_\alpha : I \to \mathfrak{g}$ by

$$q_\alpha(t) = \begin{cases} dL_{\alpha(t)^{-1}} \left(\frac{\alpha'(t)}{\sqrt{|\alpha'(t)|}} \right) & \text{if } \alpha'(t) \neq 0 \\ 0 & \text{if } \alpha'(t) = 0 \end{cases}$$

Since α is absolutely continuous, $q_\alpha \in L^2(I, \mathfrak{g})$.

The Bijection

$\alpha \mapsto (\alpha(0), q_\alpha)$ defines a bijection

$$AC(I, G) \longleftrightarrow G \times L^2(I, \mathfrak{g})$$

Riemannian Structure

We have defined a bijection

$$AC(I, G) \longleftrightarrow G \times L^2(I, \mathfrak{g}).$$

The space on the right is a product of a finite dimensional Riemannian manifold and the Hilbert space $L^2(I, \mathfrak{g})$. As a result, we have a Hilbert manifold structure on $G \times L^2(I, \mathfrak{g})$, which we can transfer to $AC(I, G)$ via the bijection. Furthermore, this Hilbert manifold structure is invariant under reparametrization and under left translation by elements of G.

Geodesics

If we have an explicit formula for geodesics on G, then we immediately know the geodesics on $AC(I, G)$, since the geodesics in $L^2(I, \mathfrak{g})$ are simply straight lines!

For many common Lie groups, we do have explicit formulae for geodesics under a left-invariant metric. Examples include all compact matrix groups, $SL(\mathbb{R}, N)$ and $GL(\mathbb{R}, N)$.

Modding out by Reparametrization and by Left Action of G

The group G and the reparametrization group Γ both act by isometries on $G \times L^2(I, \mathfrak{g})$. Note:

- The action of G only affects the first factor. Hence, to mod out by G, simply left translate one of the paths to make the two starting points of the paths coincide!
- The action of Γ only affects the second factor. Since $\mathfrak{g}$ is a finite dimensional inner product space, we may identify $\mathfrak{g} \equiv \mathbb{R}^N$, and then use the dynamic programming algorithms we have already programmed, essentially without alteration, to find the optimal registration between two paths.

Summing up the Lie Group approach

- This approach is a generalization of the SRVF (In fact, $\mathbb{R}^N$ is a Lie group under addition.)
- Geodesics between parametrized curves are usually faster to compute than in the Hilbert Bundle Approach. This is because left translation is easier to compute than parallel translation!
- Registration is nearly as easy as the SRVF (dynamic programming).
- There is no arbitrary choice of reference point, hence no distortion for curves that stray far from a reference point.
- This metric is invariant under left translation in G.

Homogeneous Spaces

We now turn to work by Su et al.[3,4] on **homogeneous spaces**. It starts with the same construction as the previously cited work of Celledoni et al., but adds a twisting construction to capture the topological non-triviality of the tangent bundle of the homogeneous space. This makes it possible to analyze arbitrary sets of AC curves in a homogeneous space, instead of just sets of curves that all start at the same point. This work was done independently and at almost the same time as the work of Celledoni et al. on homogeneous spaces.

[3] Z. Su, E. Klassen, and M. Bauer, *The Square Root Velocity Framework for Curves in a Homogeneous Space*, 8 pages, CVPR, July, 2017.

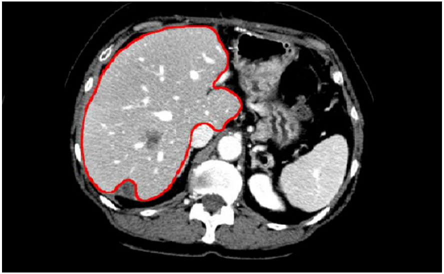
[4] Z. Su, E. Klassen, and M. Bauer, *Comparing Curves in Homogeneous Spaces*, Differential Geometry and its Applications, 2018.

Homogeneous Spaces

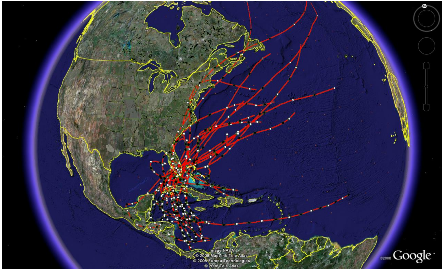
Many (if not most) Riemannian manifolds appearing in applications can be viewed as **homogeneous spaces**. For example:

- Euclidean spaces $\mathbb{R}^n$
- spheres S^n
- hyperbolic spaces $\mathbb{H}^n$
- Grassmannian manifolds
- Stiefel manifolds
- the space of $n \times n$ positive definite symmetric matrices (PDSM)

Curves in Homogeneous Spaces



Curves in $\mathbb{R}^2$



Curves on S^2

Definition of Homogeneous Space

In this presentation, we define a *homogeneous space* to be a quotient

$$M = G/K,$$

where G is a finite dimensional Lie group and K is a compact Lie subgroup. Let $\pi : G \to M$ denote the quotient map. G acts transitively on M from the left by $g * (hK) = (gh)K$.

Let

$$\mathfrak{g} = \text{ the Lie algebra of } G = T_1 G$$

$$\mathfrak{k} = \text{ the Lie algebra of } K = T_1 K$$

Examples:

- Sphere: $S^n = SO(n+1)/SO(n)$
- Grassmannian: $G(n,d) = O(n)/(O(d) \times O(n-d))$
- Hyperbolic space: $H^n = SO(n,1)/S(n)$

Riemannian Metric on M

In this situation, we can always endow G with a Riemannian metric that is left-invariant under multiplication by G and bi-invariant under multiplication by K. This metric induces a metric on the quotient $M = G/K$ that is invariant under the left action of G.

We could not construct such a metric if we did not assume that K was compact.

Absolutely continuous curves in M

Denote by $AC(I, M)$ the set of absolutely continuous curves $\beta : I \to M$, where $I = [0, 1]$.

Main goal: Put a smooth structure and a Riemannian metric on $AC(I, M)$ that is invariant under the left action of G and also under the right action of the reparametrization group $\Gamma = Diff_+(I, I)$.

Method: Contruct a bijection between $AC(I, M)$ and a space that can easily be endowed with these structures. We now state this bijection.

Main Bijection

Let $\mathfrak{k}^\perp$ denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$.

Theorem

There is a bijection

$$\Phi : (G \times L^2(I, \mathfrak{k}^\perp))/K \to AC(I, M),$$

which we will define in the following slides. We will also define a Hilbert manifold structure and a Riemannian metric on $(G \times L^2(I, \mathfrak{k}^\perp))/K$, which yield the desired structures on $AC(I, M)$. These structures are preserved by the action of the reparametrization group Γ and by the left action of G.

Building the Bijection: First Step

Following the SRVF, we define a bijection

$$Q : AC(I, G) \to G \times L^2(I, \mathfrak{g})$$
$$Q(\alpha) = (\alpha(0), q),$$

where

$$q(t) = \begin{cases} L_{\alpha(t)^{-1}} \dfrac{\alpha'(t)}{\sqrt{||\alpha'(t)||}} & \alpha'(t) \neq 0 \\ 0 & \alpha'(t) = 0 \end{cases}$$

where $L_{\alpha(t)^{-1}}$ denotes left translation by $\alpha(t)^{-1}$ in the Lie group G.

Building the Bijection: Second Step

Let $AC^{\perp}(I, G)$ denote the set of absolutely continuous curves in G that are perpendicular to each coset gK that they meet (i.e., "horizontal curves"). Then it is immediate that Q induces a bijection

$$AC^{\perp}(I, G) \to G \times L^2(I, \mathfrak{k}^{\perp}).$$

Building the Bijection: Third Step

Theorem

Given $\beta \in AC(I, M)$ *and* $\alpha_0 \in \pi^{-1}(\beta(0))$, *there is a unique horizontal lift* $\alpha \in AC^{\perp}(I, G)$ *satisfying* $\beta = \pi \circ \alpha$ *and* $\alpha(0) = \alpha_0$.

Because the right action of K on G preserves $AC^{\perp}(I, G)$ and acts freely and transitively on $\pi^{-1}(\beta(0))$, we obtain the following:

Theorem

π *induces a bijection*

$$AC^{\perp}(I, G)/K \to AC(I, M).$$

Main Bijection proved!

The bijections on the last two slides imply the main bijection:

$$\Phi : (G \times L^2(I, \mathfrak{k}^\perp))/K \to AC(I, M).$$

The formula for the K-action appearing on the left side of this bijection is

$$(g, q) * y = (gy, y^{-1}qy),$$

where $g \in G$, $q \in L^2(I, \mathfrak{k}^\perp)$, and $y \in K$.

Smooth structure and Riemannian metric

$L^2(I, \mathfrak{k}^\perp)$ is a Hilbert space; hence it is a Hilbert manifold with the obvious Riemannian metric.

G has already been given the structure of a finite dimensional Riemannian manifold.

Hence, $G \times L^2(I, \mathfrak{k}^\perp)$ is a Riemannian Hilbert manifold.

Since K is a compact Lie group acting freely by isometries, it follows that the quotient map induces a Riemannian metric on

$$(G \times L^2(I, \mathfrak{k}^\perp))/K,$$

making it into a Riemannian Hilbert manifold.

Geodesics and distance

A geodesic in $G \times L^2(I, \mathfrak{k}^\perp)$ is the product of a geodesic in G with a straight line in $L^2(I, \mathfrak{k}^\perp)$, making geodesics in this product space easy to compute. The length of such a geodesic is computed from the lengths of its two factors by the Pythagorean Theorem.

A geodesic in $(G \times L^2(I, \mathfrak{k}^\perp))/K$ is the image of a geodesic in $G \times L^2(I, \mathfrak{k}^\perp)$ that is perpendicular to the K-orbits that it meets.

Method for obtaining geodesics in $AC(I, M)$

Given elements β_0 and β_1 in $AC(I, M)$, we identify them via Φ with the corresponding K-orbits in $G \times L^2(I, \mathfrak{k}^\perp)$; denote these orbits by $[\alpha_0, q_0]$ and $[\alpha_1, q_1]$.

1. Determine $y \in K$ that minimizes

$$d((\alpha_0, q_0), (\alpha_1, q_1) * y).$$

   Because K is a compact Lie group, this optimization is generally straightforward using a gradient search.

2. Calculate the geodesic in $G \times L^2(I, \mathfrak{k}^\perp)$ from (α_0, q_0) to $(\alpha_1, q_1) * y$. –Easy, if we know the geodesics in G (which we often do).
3. Under the above correspondences, this geodesic will give a shortest geodesic between β_0 and β_1 in $AC(I, M)$.

Unparametrized Curves, or "Shapes," in M

Define the *shape space*, or the space of unparametrized curves, by

$$\mathcal{S}(I, M) = AC(I, M)/\sim$$

where

$$\beta_1 \sim \beta_2 \Leftrightarrow \mathrm{Cl}(\beta_1 \Gamma) = \mathrm{Cl}(\beta_2 \Gamma)$$

Recall that $\Gamma = \mathit{Diff}_+(I)$; closure is with respect to the metric we have put on $AC(I, M)$. The distance function on $\mathcal{S}(I, M)$ is given by

$$d([\beta_1], [\beta_2]) = \inf_{\gamma_1, \gamma_2 \in \bar{\Gamma}} d(\beta_1 \circ \gamma_1, \beta_2 \circ \gamma_2)$$

where $\bar{\Gamma} =$

$$\{\gamma \in AC(I, I) : \gamma'(t) \geq 0, \gamma(0) = 0, \gamma(1) = 1\}.$$

Existence of Optimal Reparametrizations

For $M = \mathbb{R}^n$, the existence of optimal reparametrizations $\gamma_1, \gamma_2 \in \bar{\Gamma}$ has been proved in the following two cases:

- If at least one of the curves β_1 or β_2 is PL: proved by Lahiri, Robinson and Klassen in 2015.
- If both β_1 and β_2 are C^1: proved by M. Bruveris in 2016.

Both of these theorems have easy generalizations to the case of an arbitrary homogeneous space M.

Example: the Sphere S^n

S^n is a homogeneous space since

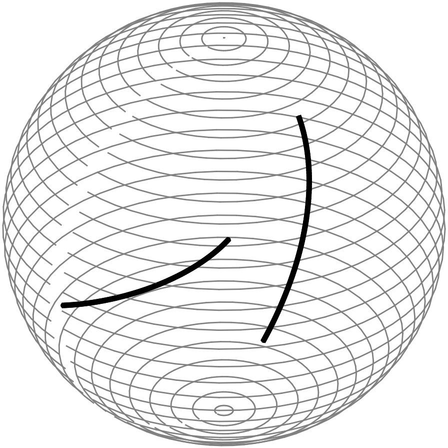
$$S^n \cong SO(n+1)/SO(n)$$

We use the usual metric on $SO(n+1)$:

$$\langle u, v \rangle_g = \mathrm{tr}(u^t v),$$

which has the required invariance properties.

Geodesic Between Two Unparametrized Curves on S^2



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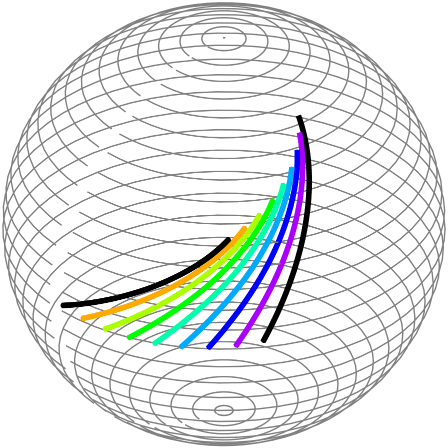
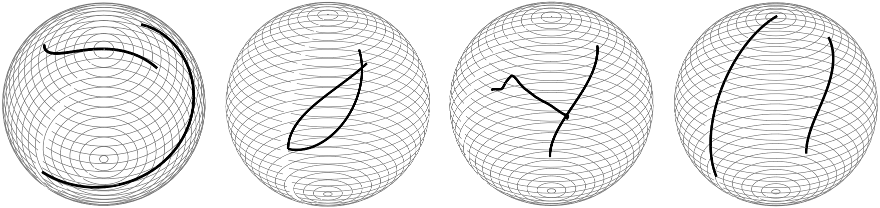


More Examples on S^2



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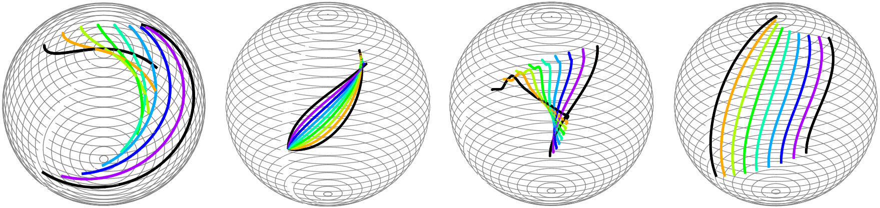


Applications to Hurricane Tracks

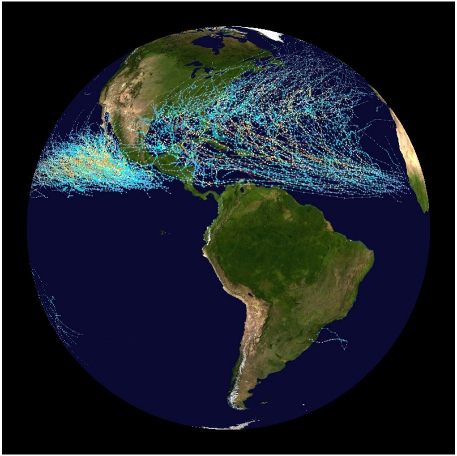


Figure: Hurricane tracks cumulative from 1950 to 2005 obtained from the National Hurricane Center website.

Multi-dimensional Scaling

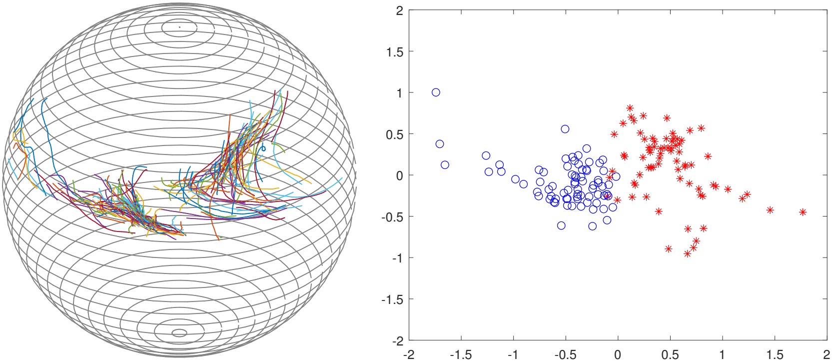


Figure: Left: 75 hurricane tracks from the Atlantic hurricane database and 75 hurricane tracks from the Northeast and North Central Pacific hurricane database. Right: multi-dimensional scaling in two dimensions. Atlantic $\star$; Pacific $\circ$.

Karcher Means

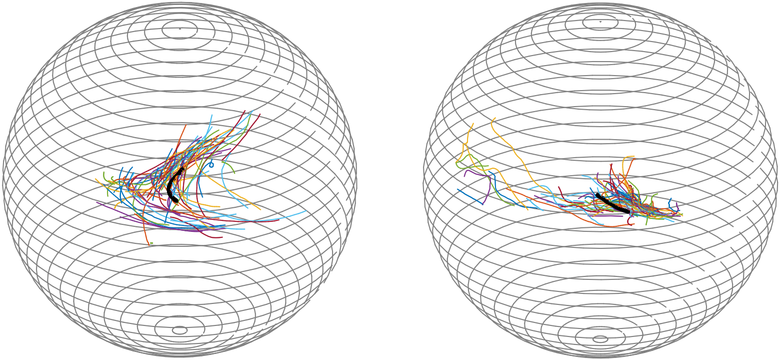
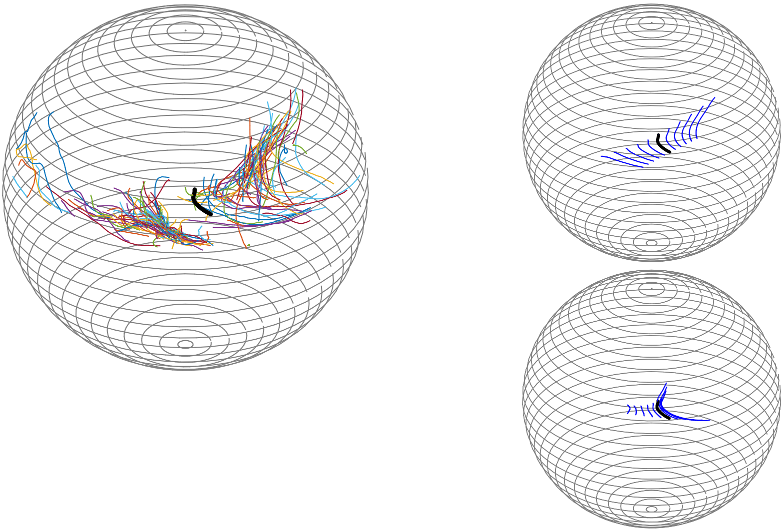


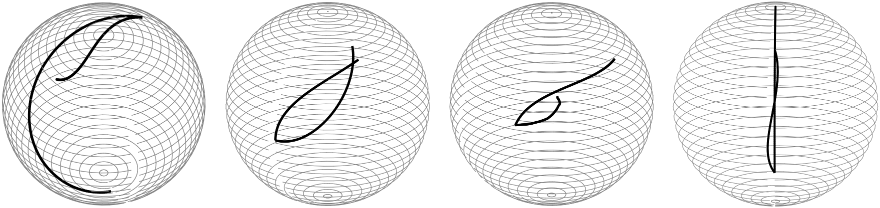
Figure: The Karcher means (up to reparametrization) of 75 hurricane paths in the Atlantic (left) and Pacific (right).

The First Two Principal Directions



Geodesics in the Space of Unparametrized Curves Modulo Rigid Motions

$SO(3)\,(=G)$ acts on $S^2\,(=M)$ as its group of rigid motions. We can compute geodesics in the corresponding quotient space:



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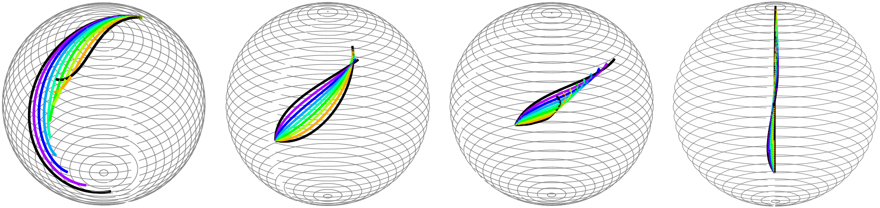

Summary:

- We have adapted the SRVF to compare curves in a homogeneous space.
- This method is usually more computationally efficient than the Hilbert Bundle method.
- This method does not have the main drawback of TSRVF: there is no arbitrary reference point and, hence, no distortion. Also this metric is invariant under the left action of G on $M = G/K$.

One Common Feature of Methods Shown So Far

In all the methods so far, we transport the velocity vectors of a curve to a single point in order to compare them and calculate geodesics. This point is either:

- An arbitrary reference point.
- The starting point of the curve.
- The identity point of a Lie group.

This strategy makes sense for manifolds that are "homogeneous" in the sense that they have similar curvature at all points. It also makes computation more efficient.

Manifolds With Irregular Curvature

For manifolds with varying curvature (e.g., if M is *not* homogeneous), it can be argued that these methods won't "see" the irregularities of the manifold that different parts of the trajectories pass through.

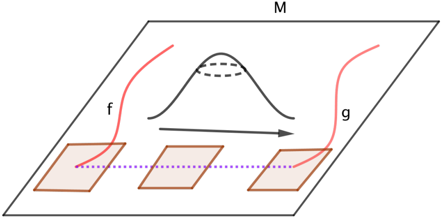

Figure: As f is deformed to g, the middle of the curve must negotiate a "bump." However, the path of tangent spaces at the starting points of the deformed curves will not "see" this bump.

Le Brigant's Method

Alice Le Brigant has defined[5] a metric on trajectories in a Riemannian manifold M, also modeled after the SRVF, that will take into account the local structure of M at all points that a deformed trajectory passes through; not just at a reference point, and not just along the initial points of the trajectories.

While Le Brigant's method might theoretically be the most comprehensive generalization of the SRVF to an arbitrary Riemannian manifold, its computational cost is rather large and, thus, limits its usefulness for large data sets.

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[5] A. Le Brigant, *Computing distances and geodesics between manifold-valued curves in the SRV framework*, J. Geom. Mech. 9(2):131-156, 2017.

Le Brigant's Metric on $C^\infty(I, M)$

Like the original SRVF for $\mathbb{R}^N$, Le Brigant's metric is a 1st-order Sobolev metric and is reparametrization-invariant; also like the SRVF, there are two ways to formulate Le Brigant's metric:

1. As an elastic metric, penalizing "bending" and "stretching" of the velocity vectors, using the same two coefficients as the SRVF.
2. As a pulled-back metric, using an L^2 type metric on a certain representation space for these trajectories.

To make this precise, let $Imm(I, M)$ denote the set of C^∞ immersions $f : I \to M$, i.e., we are assuming that f is C^∞ and $f'(t) \neq 0$ for all $t \in I$.

(1) Elastic Metric Formulation

Given $f \in Imm(I, M)$, let w and z be two smooth vector fields (tangent to M) along f. We can think of w and z as elements of $T_f Imm(I, M)$.

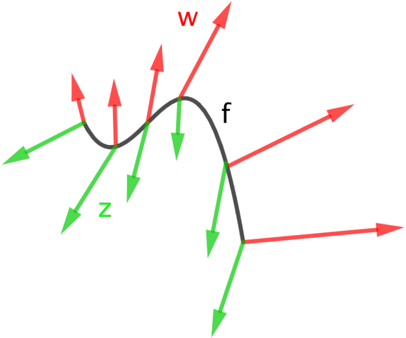

Figure: Two tangent vector fields, w and z, along the trajectory f

Define a Riemannian metric on $Imm(I, M)$ by

$$G_f(w, z) = \langle w(0), z(0)\rangle + \int_f \langle \nabla_s w^N, \nabla_s z^N\rangle + \frac{1}{4}\langle \nabla_s w^T, \nabla_s z^T\rangle ds,$$

where the integral is taken along the trajectory f with respect to arclength, $\nabla_s w$ denotes the covariant derivative of w along f with respect to arclength, and the superscripts N and T mean to take the components normal to f' and tangent to f', respectively.

Note that this formula is very similar to the formula for the elastic metric used in the formulation of SRVF for curves in $\mathbb{R}^N$; the only difference is the necessity to use covariant differentiation in the Riemannian manifold M.

(2) Representation Space Formulation

This metric can also be expressed in terms of an SRVF-type representation of the trajectories. Let TM denote the total space of the tangent bundle of M and define an embedding

$$Q : Imm(I, M) \to C^\infty(I, TM)$$

by

$$Q(f)(t) = \frac{f'(t)}{\sqrt{|f'(t)|}}.$$

Le Brigant observes that if we put a natural metric on $C^\infty(I, TM)$ related to the Sasaki metric, and pull this metric back to $Imm(I, M)$ using Q, we obtain the same metric as the elastic metric defined on the previous slide!

Why Is This More Complicated than the $\mathbb{R}^N$ Case?

Two reasons:

- Unlike $L^2(I, \mathbb{R}^N)$, $C^\infty(I, TM)$ is not a linear space, so it is not easy to compute geodesics.
- Unlike the SRVF in the case of $\mathbb{R}^N$, the map Q is far from being a bijection! So, even if we can compute geodesics in $C^\infty(I, TM)$, they generally do not lie in the image of Q and, hence, do not provide geodesics in $Imm(I, M)$.

In conclusion, several operations that are straightforward for $AC(I, \mathbb{R}^N)$ are computationally expensive for Le Brigant's metric on $Imm(I, M)$:

- Computing the geodesic between two parametrized trajectories.
- Computing the exponential map on $Imm(I, M)$.
- Computing optimal registrations between trajectories in M.
- $Imm(I, M)$ is not geodesically complete, because of the possible vanishing of velocity vectors. In the case of SRVF for $\mathbb{R}^N$, this problem was solved by including all absolutely continuous curves, but it is not clear how to extend Le Brigant's metric to make this work for arbitrary M.

End

Thank You