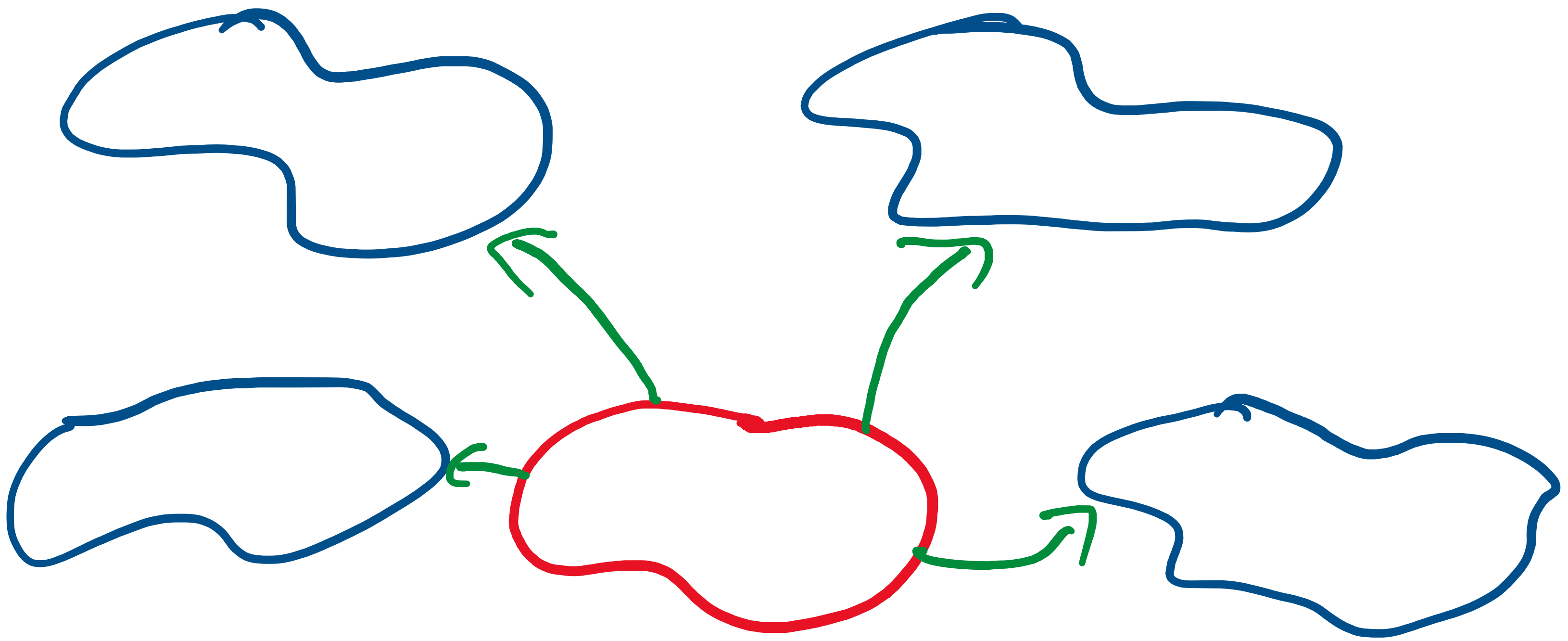
Introduction to Metric Registration.

Diffeomorphic registration



Registration : estimate a transformation
aligning a template with a target

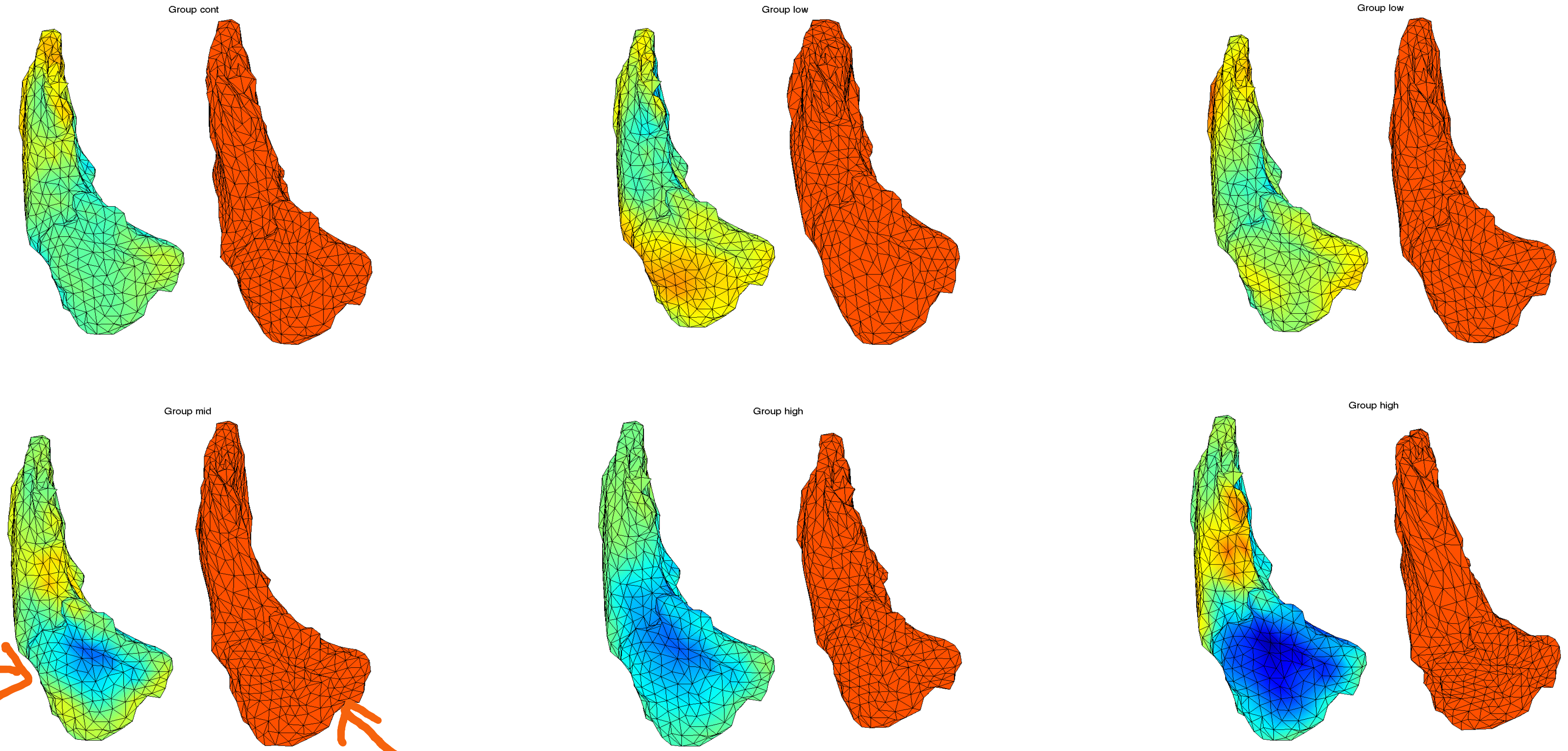
→ Interest: provide a new representation (coordinates) of the target.

→ Let m_0 denote the template.

Registration

$$m \longmapsto \varphi_m$$

φ_m : diffeomorphism such that $\varphi_m(m_0) \simeq m$



Representation of the registration diffeomorphism on the template.

Target shape

Caudates from PREDICT-HD (Huntington disease)

→ **Recall**: If $\Omega \subset \mathbb{R}^d$ (open)

A diffeomorphism of Ω is a differentiable one-to-one transformation from Ω onto itself, with a differentiable inverse.

→ Denote by $\mathcal{R}$ (or $\mathcal{R}_{m_0}$) the registration operation : $m \longmapsto \varphi_m = \mathcal{R}(m)$

More notation.

- $\mathrm{Diff}_0(\mathbb{R}^d)$: diffeomorphisms of $\mathbb{R}^d$ such that φ, φ^{-1} tend to id at ∞.
- $\mathrm{Diff}_0^p(\mathbb{R}^d)$: C^p diffeomorphisms.
- $\mathcal{M}$: shape space, open subset of Banach space Q.
- Assume a group action of Diff_0^p on $\mathcal{M}$

$$m \longmapsto \varphi \cdot m$$

A group action (on the left) is an operation $(\varphi, m) \mapsto \varphi . m$ from $\mathrm{Diff} \times M$ to M such that

$$\varphi . (\psi . m) = (\varphi \circ \psi) . m$$

A registration algorithm is a mapping :

$$\mathcal{R}_{m_0} : \mathcal{M} \longrightarrow \mathrm{Diff}$$

such that $\mathcal{R}_{m_0}(m) \cdot m_0 = m$

Diffeomorphism

or $\simeq$

There is a large number of registration methods in the literature (not necessarily diffeomorphic).

See work, e.g., from Thirion, Christensen-Miller-Rabbit, Guimond, Vercauteren, Pennec, Benamou-Brenier, Tannenbaum (images); Yau-Gu-Ye, Memoli, Styner (surfaces); Klassen, Srivastava, Kurtek... (curves) etc.

We here focus on a small subset: Metric Registration

Registration is under constrained.

- Define $\pi : \mathrm{Diff} \longrightarrow \mathcal{M}$

$$\varphi \longmapsto \varphi \cdot m_0$$

- For $m \in \mathcal{M}$, define the "fiber"

$$F_m = \pi^{-1}(m) = \{\varphi : \pi(\varphi) = m\}$$

- We have: $\varphi, \psi \in F_m \Longleftrightarrow (\varphi^{-1} \circ \psi) m_0 = m_0$

i.e., $\varphi^{-1} \circ \psi \in \sigma(m_0)$, the stabilizer of $m_0 \in \mathrm{Diff}$

$$\sigma(m_0) = \{\varphi : \varphi \cdot m_0 = m_0\}$$

- If $\sigma(m_0) \neq \{id\}$: $\varphi \cdot m_0 = m$ is ambiguous as a constraint for registration.

- Registration algorithms often use this constraint as part of an optimization problem

$$\begin{cases} R_{m_0}(m) = \operatorname{argmin}\left(\operatorname{cost}(\varphi)\right) \\ \text{subject to } \varphi \cdot m_0 = m . \end{cases}$$

Metric registration requires that:

- $\mathrm{cost}(\varphi) = d(\mathrm{id}, \varphi)$
  (d a distance on Diff)

- The optimal cost is also a distance between m_0 and m.

Metrics and submersions

- Consider sets G and B and a mapping

$$\pi : G \longrightarrow B$$

such that π is a surjection (i.e., it is onto).

- To $b \in B$, associate the "fiber"

$$F_b = \pi^{-1}(b) \subset G$$

- Assume that G is a metric space with distance d_G. Define:

$$d_B(b,b') = \inf\{d(g,g'),\ \pi(g)=b,\ \pi(g')=b'\}$$
$$= d_G(F_b, F_{b'})$$

Is d_B a distance on B?

Proposition If the fibers are closed and parallel, then d_B is a distance.

[Fibers are parallel if and only if, $\forall\, b' \in B$, the mapping : $g \longmapsto d_G(g, F_{b'})$ is constant over fibers]

Sub-riemannian Metrics

Assume that G is a differential manifold

for example: G open subset of a Banach space.

Example: $\mathrm{Diff}_0^p(\mathbb{R}^d)$ is an open subset of

$$id + \underbrace{C_0^p(\mathbb{R}^d, \mathbb{R}^d)}_{\text{C^p vector fields vanishing at infinity}}$$

A <u>sub-riemannian</u> structure on G consists in

→ a collection $(D_g, g \in G)$ such that D_g is a subspace of $T_g G$.

→ a family of <u>inner products</u> $\langle\ ,\ \rangle_g$ on D_g, with associated norm denoted $\|\ \|_g$.

(We also need $(D_g, \langle\ ,\ \rangle_g)$ to depend smoothly on g)

[$(D_g, g \in G)$ is called a <u>distribution</u>

Sub-riemannian distance:

$$d_G(g,g') = \inf \sqrt{\int_0^1 \|\dot{\gamma}(t)\|_m^2 \, dt}$$

The minimum is over all paths $\gamma : [0,1] \to G$

such that
$$\begin{cases} \gamma(0) = g \,, \quad \gamma(1) = g' \,, \\ \dot{\gamma}(t) \in D_{\gamma(t)} \,, \quad \forall t \,, \\ \gamma \text{ continuous, piecewise differentiable.} \end{cases}$$

→ A point g' is attainable from g if there exists a path with finite energy between them.

→ $d_G(g,g') = \infty$ if g' is not attainable from g.

→ d_G is always symmetric and satisfies the triangle inequality.

Subriemannian submersion

→ Assume a subriemannian structure on G.

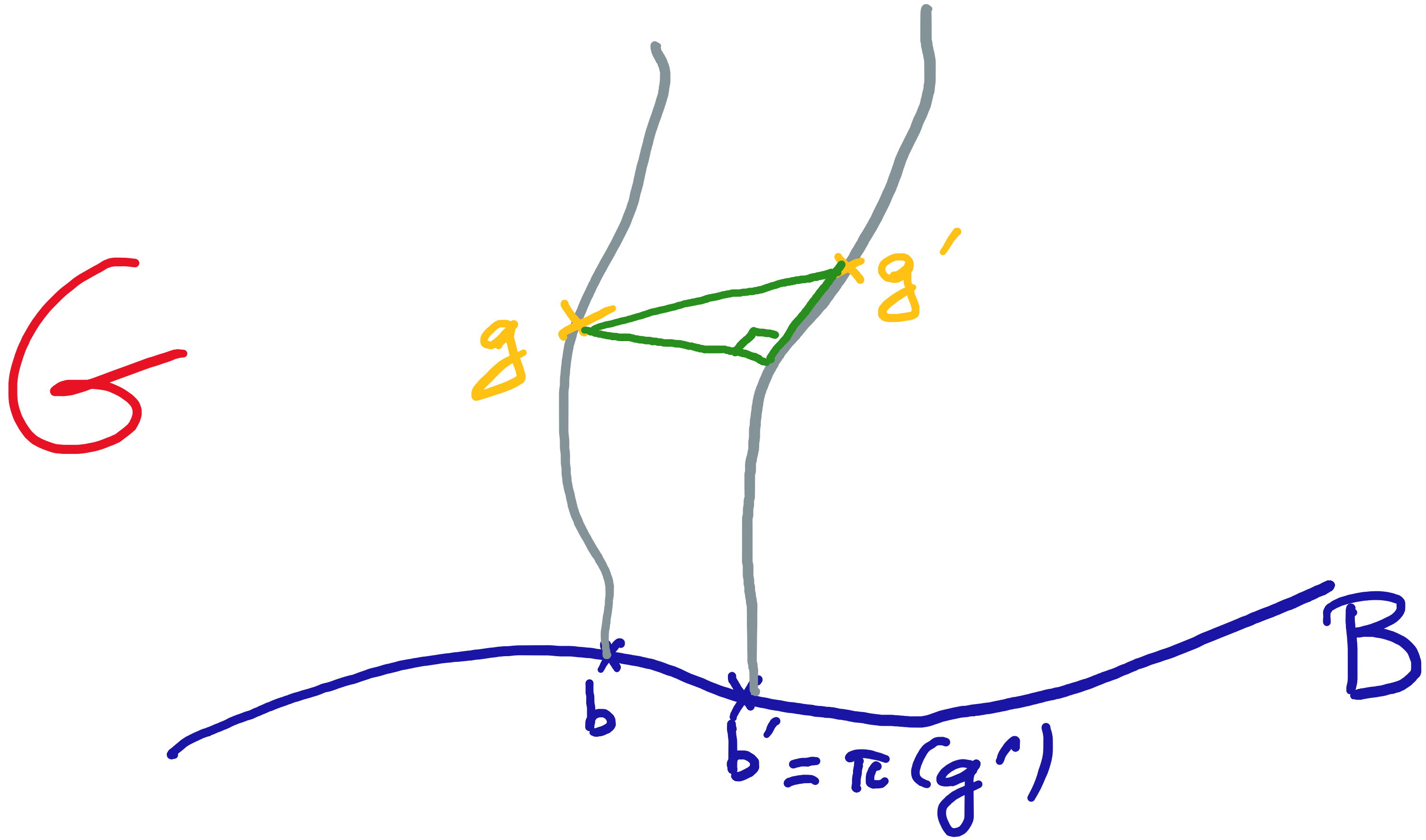
→ Let B be a second differential manifold.

→ Let $\pi : G \to B$ be a submersion.

[i.e., π is C^1, onto and $d\pi(g)$ onto for all $g \in G$.]

→ Define $d_B(b, b') = d_G(F_b, F_{b'})$

with $F_b = \pi^{-1}(b)$: fiber.


G
g
g'
b
b' = π(g')
B

→ A path $\beta : [0,1] \to B$ is admissible if it can be lifted to an admissible path $\gamma : [0,1] \to G$, i.e.,

$$\beta(t) = \pi(\gamma(t)).$$

→ If β is admissible, then $\dot{\beta} = d\pi(\gamma)\,\dot{\gamma}$, which implies that

$$\left[\ \dot{\beta} \in d\pi(\gamma)\, D_\gamma \text{ at all times.}\right.$$

→ For consistency, we require $\underline{d\pi(g) D_g = d\pi(g') D_{g'} \text{ if } \pi(g) = \pi(g')}$.

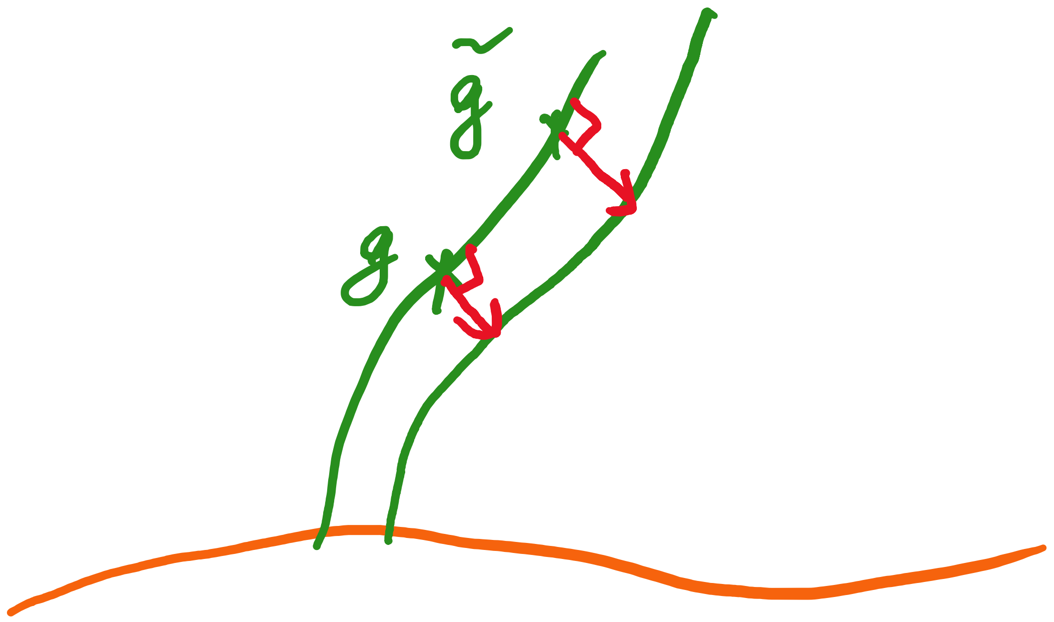
→ One can then define the space $\underline{\Delta_b = d\pi(g) D_g}$, $g \in F_b$ without ambiguity.

$\longrightarrow$ Vertical space at g : $V_g = \text{Null}(d\pi(g))$

$\longrightarrow$ Define $H_g = (V_g \cap D_g)^{\perp}$: horizontal space

Then $d\pi(g)$ is one-to-one from H_g onto $W_{\pi(g)}$

$\Longrightarrow$ the horizontal spaces along the same fiber are in one-to-one correspondence.



→ Distances between nearby fibers are measured by horizontal vectors.

→ <u>For parallelism</u>: Correspondences between horizontal spaces must be <u>isometries</u> (i.e., length preserving).

A class of distributions in $\mathrm{Diff}^p_0(\mathbb{R}^d)$

→ If $G = \mathrm{Diff}^p_0(\mathbb{R}^d)$, $T_\varphi G \simeq C^p_0(\mathbb{R}^d, \mathbb{R}^d)$

→ Let V be a Hilbert space continuously included in $C^p_0(\mathbb{R}^d, \mathbb{R}^d)$, dense in that space.

Define $\mathcal{D}_\varphi = \{ v \circ \varphi \,,\ v \in V \}$

with $\| v \circ \varphi \|_\varphi = \| v \|_V$

→ Let d_V denote the associated sub-riemannian distance.

→ We have $d_V(\psi, \tilde{\psi}) = \inf_v \sqrt{\int_0^1 \|v(t)\|_V^2 \, dt}$

over all time-dependent vector fields v s.t. $\varphi_{01}^v \circ \psi = \tilde{\psi}$

Here φ_{st}^v is the flow associated to v, s.t.

$$\left[\partial_t \varphi_{st}^v = v(t, \varphi_{st}^v) \text{ and } \varphi_{ss}^v = id \right.$$

Properties of the distance

(1) Recall: $\varphi \in \mathrm{Diff}_0^p$ is attainable if

$$d_V(id, \varphi) < \infty$$

The set of attainable diffeomorphisms is a subgroup of $\mathrm{Diff}_0^p(\mathbb{R}^d)$, denoted Diff_V

(2) $(\mathrm{Diff}_{\mathbb{V}}, d_{\mathbb{V}})$ is a complete metric space.

(3) $\forall \varphi \in \mathrm{Diff}_{\mathbb{V}}$, $\exists v \in L^2([0,1], \mathbb{V})$

such that

$$d_{\mathbb{V}}(id, \varphi)^2 = \int_0^1 \|v(t)\|_{\mathbb{V}}^2 \, dt$$

(See Trouvé; Dupuis - Grenander - Miller)

Choosing $\mathbb{V}$

→ **Assumption:** $\mathbb{V}$ is a Hilbert space of vector fields on $\mathbb{R}^d$, continuously embedded in $C_0^p(\mathbb{R}^d, \mathbb{R}^d)$. $(p \geq 1)$

i.e., $\exists\, c$ s.t., $\forall v \in \mathbb{V}$

$$\|v\|_{p,\infty} \leq c\, \|v\|_{\mathbb{V}}$$

→ This implies that $\mathbb{V}$ is a reproducing kernel Hilbert space :

$$\exists K : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathcal{M}_d(\mathbb{R})$$

↳ d×d matrices

s.t.,

- $\forall x, a \in \mathbb{R}^d : \left(y \longmapsto K(y,x)a\right) \in \mathbb{V}$
- $\forall v \in \mathbb{V} : \langle v, K(\cdot, x)a \rangle_{\mathbb{V}} = a^T v(x)$

→ K is positive, i.e.,

$$\forall x_1, \dots, x_n \in \mathbb{R}^d$$

$$\forall a_1, \dots, a_n \in \mathbb{R}^d$$

$$\left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_i^T K(x_i, x_j) a_j \geq 0 \right.$$

(if the x_i's are pairwise distinct, the sum vanishes only for $a_i = 0$, $\forall i$)

- Main example in applications : Matérn kernel

$$K(x,y) = P_p\left(\frac{|x-y|}{\sigma}\right) e^{-\frac{|x-y|}{\sigma}} Id$$

where P_p is a reversed Bessel polynomial of order p.

- The associated RKHS is embedded in $C_0^p(\mathbb{R}^d, \mathbb{R}^d)$.
- Other example : Gaussian kernel

$$K(x,y) = e^{-\frac{|x-y|^2}{2\sigma^2}} Id$$

Submersions onto shape spaces

Landmarks

→ Take $G = \mathrm{Diff}^p_0(\mathbb{R}^d)$

→ Let $\mathcal{M} = \{(x_1, \dots, x_N) \in \mathbb{R}^d : x_i \neq x_j \text{ if } i \neq j\}$

→ Fix $m_0 = (x_1^0, \dots, x_N^0) \in \mathcal{M}$

→ Define $\pi(\varphi) = \varphi \cdot m_0 = (\varphi(x_1^0), \dots, \varphi(x_N^0))$

Then:

→ π is onto.

→ $d\pi(\varphi)h = (h(x_1^0), \dots, h(x_N^0))$ is onto.

→ Use the previous sub-riemannian structure

$$\begin{cases} D_\varphi = \mathbb{V}\circ\varphi \\ \|v\circ\varphi\|_\varphi = \|v\|_{\mathbb{V}} \end{cases}$$

→ The vertical space at φ is

$$V_\varphi = \{h \in D_\varphi : h(x_i^0) = 0\} = \{v \in \mathbb{V} : v(x_i) = 0\} \circ \varphi$$

$$\hookrightarrow x_i := \varphi(x_i^0)$$

Let $\mathbb{V}_\varphi = \{v \in \mathbb{V} : v(\varphi(x_i^0)) = 0\} \subset \mathbb{V}$

→ We have $H_\varphi = V_\varphi^\perp \cap D_\varphi$ for the $\langle\ ,\ \rangle_\varphi$ inner product.

→ By construction $H_\varphi = \mathbb{H}_\varphi \circ \varphi$

where $\mathbb{H}_\varphi = \mathbb{V}_\varphi^\perp$ for the $\mathbb{V}$-inner product

→ H_φ is isometric to $\mathbb{H}_\varphi$ and $\mathbb{H}_\varphi$ only depends on $x = \pi(\varphi)$.

$\Rightarrow$ Horizontal spaces over a fixed $x \in M$ are isometric.

$\longrightarrow$ $$d_M(m, \tilde{m}) = \inf_{\varphi, \tilde{\varphi}} \left\{ d_G(\varphi, \tilde{\varphi}) : \pi(\varphi) = m ; \pi(\tilde{\varphi}) = \tilde{m} \right\}$$

is a distance on M.

$\longrightarrow$ It is a Riemannian metric with

$$\left[\|h\|_m = \inf \left\{ \|v\|_V : d\pi(\varphi)\, v \circ \varphi = h \right\} \right.$$
for φ s.t. $\pi(\varphi) = x$

Explicit expression of the metric

$m = (x_1, \ldots, x_N)$.

→ If $\pi(\varphi) = m$ then $d\pi(\varphi)\, v\circ\varphi = (v(x_1), \ldots, v(x_N))$

In the RKHS V: $\min\{\|v\|_V^2 : v(x_i) = h_i,\ i = 1, \ldots, N\}$

$$= h^T K(m)^{-1} h$$

where $K(m)$ is the matrix formed with $d \times d$ blocks $\left(K(x_i, x_j),\ i,j = 1, \ldots, N \right)$

More generally...

- Let $\mathrm{Diff}_0^p(\mathbb{R}^d)$ act on a space $\mathcal{M}$ of shapes.
- Assume that, for some $m_0 \in \mathcal{M}$, the mapping

$$\varphi \longmapsto \varphi \cdot m_0 \quad \text{is onto.}$$

- Let $v \cdot m$ denote the infinitesimal action of v on m

and $\xi_m v = v \cdot m$.

Infinitesimal action :

$$v \cdot m \stackrel{\circ}{=} \partial_\varepsilon \varphi_\varepsilon \cdot m \Big|_{\varepsilon=0}$$

where

$$\varphi_0 = id$$

$$\partial_\varepsilon \varphi_\varepsilon \Big|_{\varepsilon=0} = v$$

- Define a sub-riemannian structure on $\mathbb{M}$ by

$$\begin{cases} D_m = \xi_m \mathbb{V}, & m \in \mathbb{M} \\ \|h\|_m = \inf\{\|v\|_{\mathbb{V}} : \xi_m v = h\}, & h \in D_m \end{cases}$$

and the associated distance $d_{\mathbb{M}}$.

LDDMM

- Let $\mathcal{M}$ be a shape space equipped with $d_{\mathcal{M}}$ as just defined.
- Let $\Gamma : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ be a measure of discrepancy between shapes.
- Usually, $\Gamma(m, \tilde{m}) = 0$ means that m and $\tilde{m}$ can be identified up to some invariance operation, e.g., reparametrization.

- Then, LDDMM registration minimizes

$$\left[d_M(m_0, m)^2 + \Gamma(m, m_1) \right.$$

with respect to m $\underline{(m_0, m_1 \text{ being fixed})}$.

- This is equivalent to minimizing

$$\left[d_V(id, \varphi)^2 + \Gamma(\varphi . m, m_1) \right.$$

with respect to φ.

- This is also equivalent to minimizing

$$\left[\int_0^1 \|v(t)\|_V^2 \, dt \; + \; \Gamma(\varphi(1).m_0, m_1) \right.$$

subject to $\begin{cases} \partial_t \varphi(t) = v(t) \circ \varphi(t). \\ \varphi(0) = id. \end{cases}$

- This is an optimal control problem where:

→ $v \in V$ is the control,

optimal registration ← $\varphi \in \mathrm{Diff}_0^p(\mathbb{R}^d)$ is the state.

- One more equivalent formulation: minimize

$$\left[\int_0^1 \|v(t)\|_{\mathbb{V}}^2 \, dt + \Gamma(m(1), m_1)\right.$$

subject to $\begin{cases} \partial_t m(t) = v(t) \cdot m(t) \\ m(0) = m_0 \end{cases}$

- This is an optimal control problem where

$v \in \mathbb{V}$ is the control

$m \in \mathcal{M}$ is the state

- Optimal controls, $v(t)$, must be horizontal at all times : $v(t) \in \mathbb{H}_{m(t)}$.
- There is no loss of generality in adding this constraint to the minimization problem.
- Elements of $\mathbb{H}_m$ can often be parametrized, leading to reduced formulations in which this new parameter is used as control.

Example:

matching point sets: $m = (x_1, \dots, x_N)$

then

$$\left[\mathbb{H}_m = \left\{ \sum_{k=1}^{N} K(\cdot, x_k)\alpha_k \;, \alpha_1, \dots, \alpha_N \in \mathbb{R}^d \right\} \right.$$

(finite dimensional!)

One can use $\alpha_1, \dots, \alpha_N$ as controls.

Important remark

A key point in the construction is the isometry between horizontal spaces along a fiber.

Given $\varphi, \tilde{\varphi}$: $\varphi . m_0 = \tilde{\varphi} . m_0$ $(= m)$.

$$\begin{tikzcd}[column sep=large]
\mathbb{V}_{o\varphi} \supset H_\varphi \arrow[dr, leftrightarrow, "\sim"] & \\
& \mathbb{H}_\varphi = \mathbb{H}_{\tilde{\varphi}} \subset \mathbb{V} \\
\mathbb{V}_{o\tilde{\varphi}} \supset H_{\tilde{\varphi}} \arrow[ur, leftrightarrow, "\sim"'] &
\end{tikzcd}$$

if $\pi(\varphi) = \pi(\tilde{\varphi})$.

Consequence:

The construction still holds if one allows the baseline RKHS $\mathbb{V}$ to depend on fibers, i.e., make it (and its inner product) depend on shapes $m \in \mathcal{M}$.

- **New formulation:** minimize

$$\int_0^1 \|v(t)\|^2_{m(t)} \, dt + \Gamma(m(1), m_1)$$

subject to

$$\partial_t m(t) = v(t) \cdot m(t)$$
$$m(0) = m_0$$

Numerical solutions of optimal control problems.

→ Consider the problem: minimize

$$\begin{cases} \displaystyle\int_0^1 F(q,u)\,dt + G(q(1)) \\ \text{subject to} \quad \partial_t q = f(q,u), \quad q(0) = q_0 . \end{cases}$$

→ Form the control-dependent Hamiltonian:

$$H_u(p,q) = p^T f(q,u) - F(q,u) \quad (p = \text{co-state})$$

→ The differential of the objective function (considered as a function of u alone) is given by

$$\partial_u(\text{objective}) = -\partial_u H_u(p,q)$$

with

$$\begin{cases} \partial_t q = \partial_p H_u(p,q) \\ \partial_t p = -\partial_q H_u(p,q) \\ q(0) = q_0 \; ; \quad p(1) = -\partial_q G(q(1)) \end{cases}$$

Examples

→ To apply LDDMM to specific shape spaces, $\mathcal{M}$, one needs to specify

(1) <u>The action</u> $(\varphi, m) \longmapsto \varphi \cdot m$ of diffeomorphisms on shapes

(2) <u>The data attachment term</u> $\Gamma(m, m')$ comparing the deformed template to the target.

→ The function Γ must be easily computable to allow for fast evaluations in the optimization algorithm.

→ It can (and should) be used to implement invariance requirements in the registration.

Matching point sets

→ $\mathcal{M}_N = \{ \underline{x} = (x_1, \dots, x_N) \in \mathbb{R}^d \ , \ x_i \neq x_j \text{ if } i \neq j \}$

→ $\varphi \cdot \underline{x} = (\varphi(x_1), \dots, \varphi(x_N))$

→ Simplest cost function :

$$\Gamma(\underline{x}, \underline{x}') = c \sum_{i=1}^{N} |x_i - x_i'|^2$$

→ This leads to the standard "landmark" (a labelled point) matching problem.

→ This problem assumes that points are already registered to each other and the goal is to extrapolate this alignment to the whole space.

→ This is not always the case in practice.

→ Let $\mathcal{M}_* = \bigcup_{N \geq 1} \mathcal{M}_N$ be the set of collections of points of arbitrary size.

→ Assume that points are unlabeled so that point sets that differ only by a permutation must be identified.

→ We want to define a function Γ on $\mathcal{M}_* \times \mathcal{M}_*$ such that $\Gamma(\underline{x}, \underline{x}') = 0 \iff \underline{x}$ and $\underline{x}'$ only differ by a permutation.

→ Let χ be a positive (scalar) kernel on $\mathbb{R}^d$.

→ Define, for $\underline{x} = (x_1, \dots, x_N)$, $\underline{x}' = (x_1', \dots, x_{N'}')$:

$$\gamma(\underline{x}, \underline{x}') = \sum_{k=1}^{N} \sum_{k'=1}^{N'} \chi(x_k, x'_{k'}) / NN'$$

and

$$\Gamma(\underline{x}, \underline{x}') = \gamma(\underline{x}, \underline{x}) - 2\gamma(\underline{x}, \underline{x}') + \gamma(\underline{x}', \underline{x}')$$

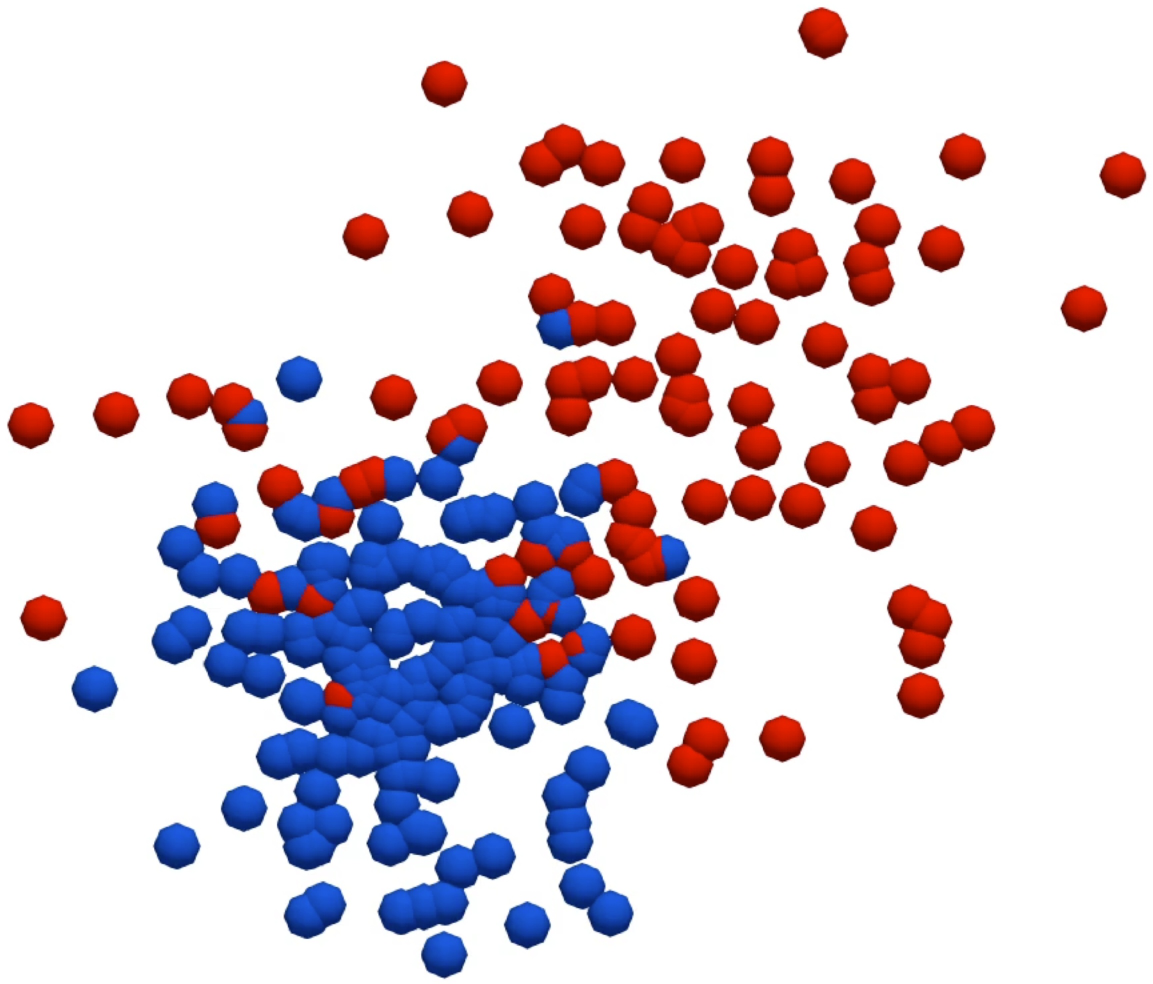
→ This defines a permutation invariant cost function between point sets.

→ Let W_X denote the RKHS associated with X

→ The function $\gamma(\underline{x}, \underline{x}')$ can be interpreted as an inner product between representations of point sets in W_X^*, given by

$$\underline{x} \longmapsto \mu_{\underline{x}} = \sum_{k=1}^{N} \delta_{x_k} / N$$

→ The representation can directly be extended to weighted point sets.



→ Similar approaches representing shapes as elements of dual spaces of RKHS's have been introduced to define reparametrization-invariant cost functions between curves, surfaces etc.

→ For example, a parametrized curve $m : [0,1] \to \mathbb{R}^d$ is represented by μ_m where

$$(\mu_m \mid f) = \int_0^1 f(m(u))\,|\dot{m}(u)|\,du \quad , \quad f \in W_X$$

→ The associated cost is

$$\Gamma(m, m') = \gamma(m,m) - 2\gamma(m,m') + \gamma(m',m')$$

with

$$\gamma(m,m') = \int_0^1\int_0^1 \chi(m(u), m'(u'))\, |\dot{m}(u)|\, |\dot{m}(u')|\, du\, du'$$

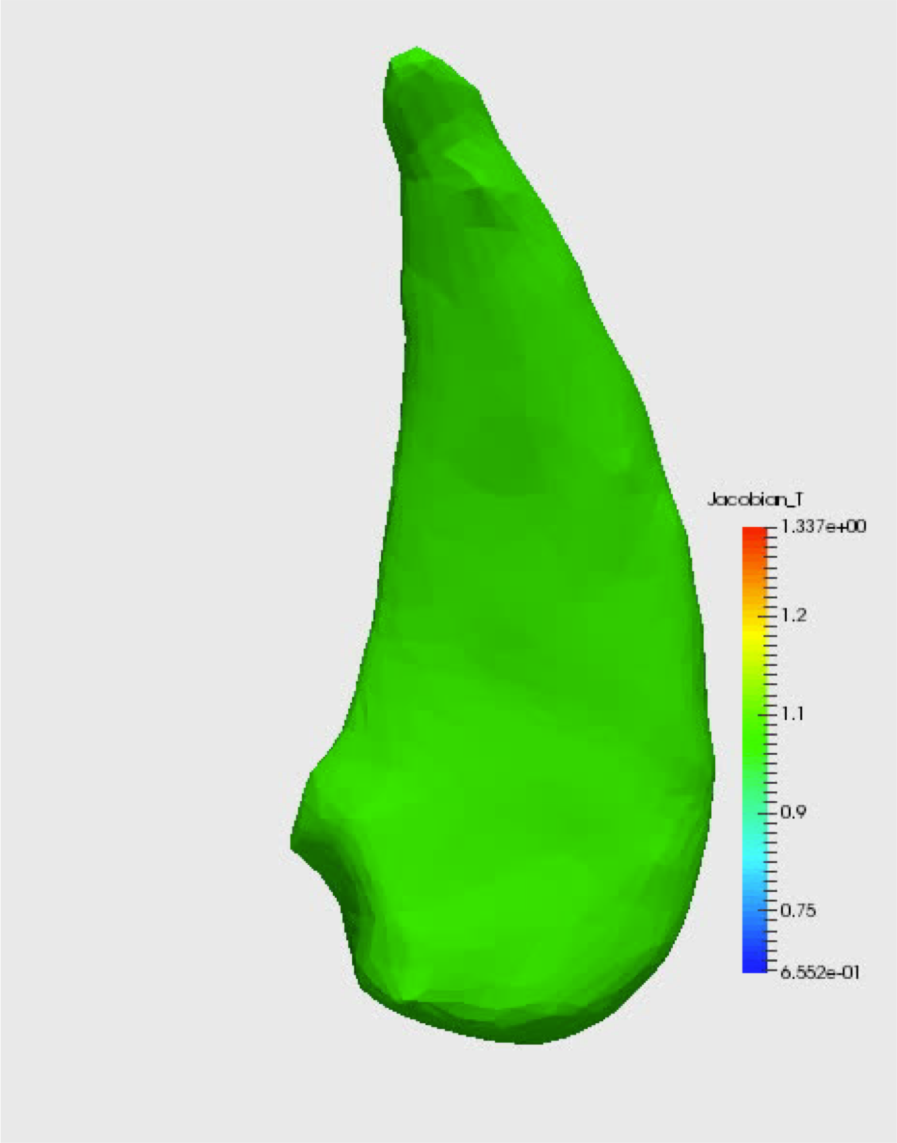
→ This is reparametrization independent: $\Gamma(m, m\circ g) = 0$ if g is a diffeomorphism of $[0,1]$.

→ This can be generalized to manifolds of arbitrary dimensions using the canonical volume.

→ One can represent shapes as linear forms over RKHSs of r-forms, where r is the dimension of the compared submanifolds of $\mathbb{R}^d$.

→ This leads to cost functions over spaces of geometric currents.

→ Shape representations as varifolds have also been introduced, leading to similar cost functions.


Jacobian_T
1.337e+00
1.2
1.1
0.9
0.75
6.552e-01

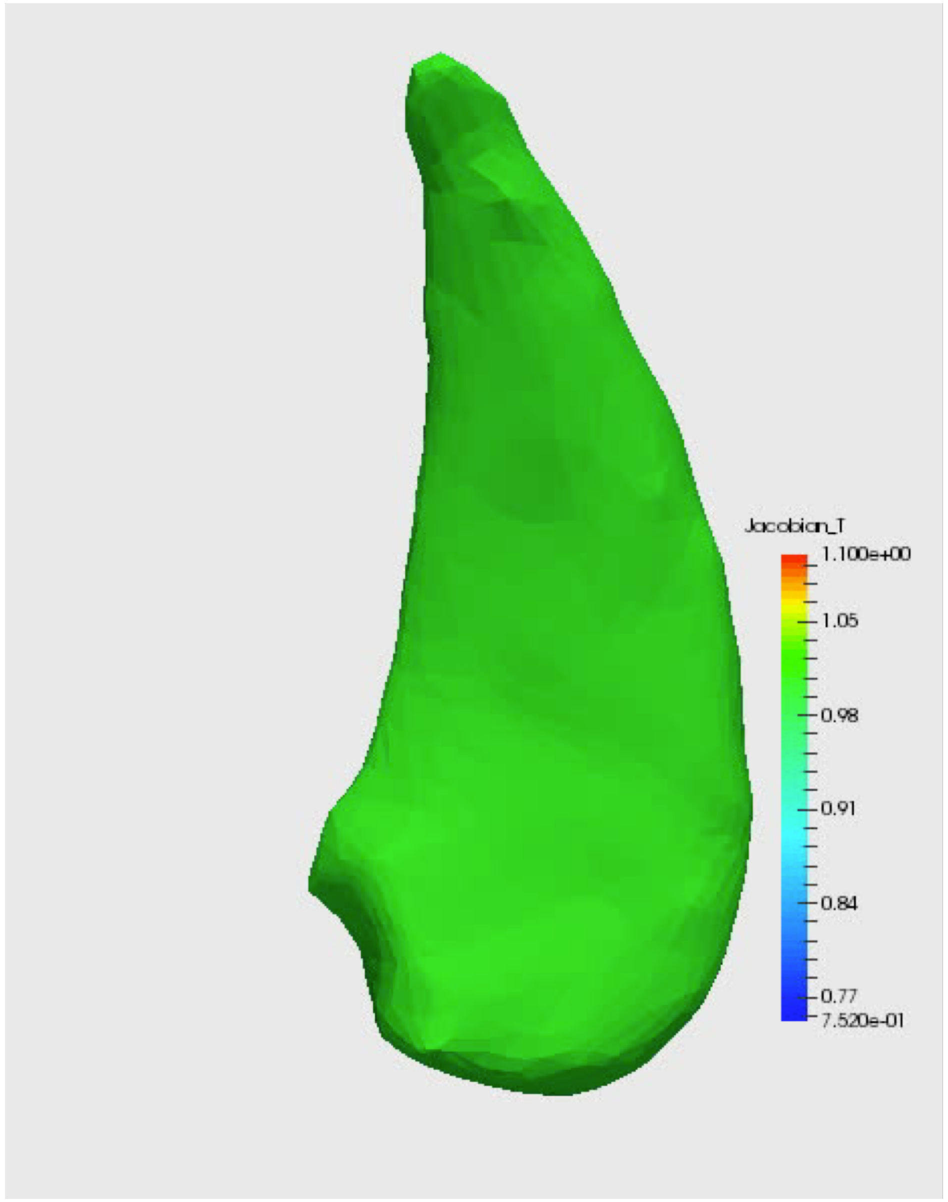

Jacobian_T
1.100e+00
1.05
0.98
0.91
0.84
0.77
7.520e-01

Image matching

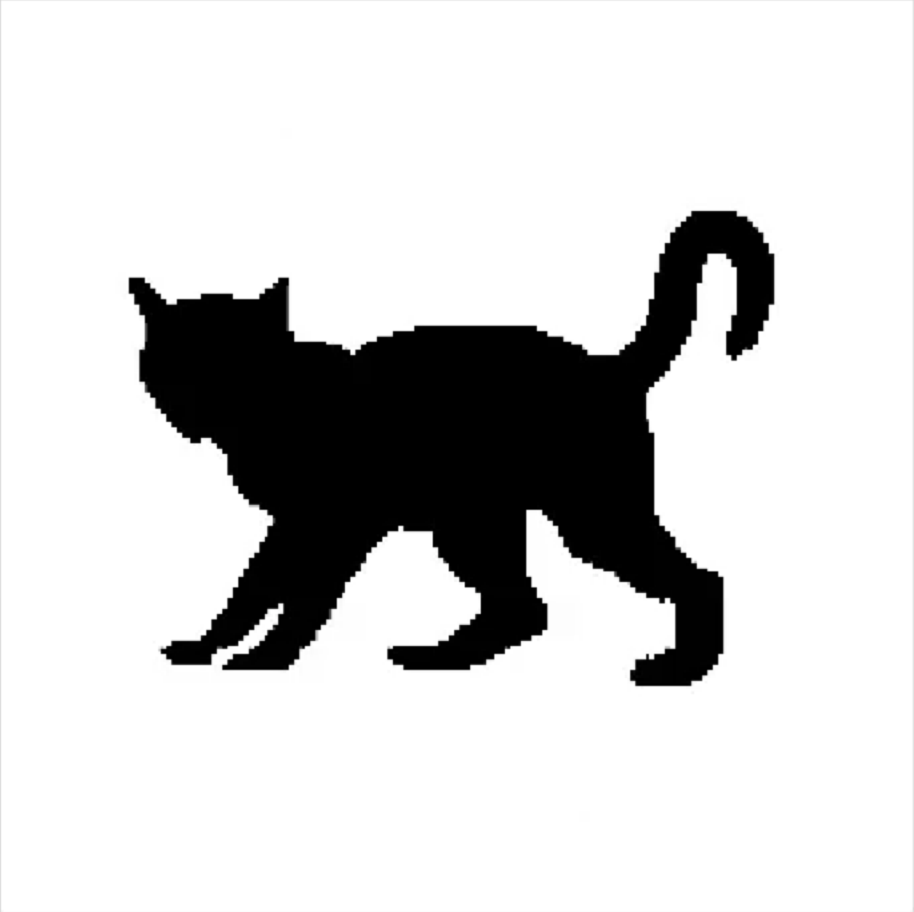
- Consider scalar images as spaces of functions

$$m : \mathbb{R}^d \longrightarrow \mathbb{R}$$

- The action of diffeomorphisms is then given by

$$\varphi \cdot m = m \circ \varphi^{-1}$$

- The quadratic cost $\Gamma(m, m') = \|m - m'\|_{L^2}^2$ leads to the original LDDMM algorithm.



Metamorphosis

→ Basic construction for LDDMM :

$$\begin{cases} G = \mathrm{Diff}_0^p(\mathbb{R}^d) \\ m_0 \in \mathcal{M} \text{ fixed} \\ \mathcal{B} = G \cdot m_0 \\ \pi(\varphi) = \varphi \cdot m_0 \end{cases}$$

Ensure that π is a Riemannian submersion

→ A single orbit $(G.m_0)$ does not always cover all shapes of interest.

→ <u>Extend</u> the construction to :

$$\begin{cases} G = \mathrm{Diff}_0^p(\mathbb{R}^d) \times \mathcal{M} \\ B = \mathcal{M} \\ \pi(\varphi, m) = \varphi \cdot m \end{cases}$$

→ This leads to <u>metamorphosis</u>

→ We need a Riemannian metric on $G \times M$ such that horizontal spaces are isometric.

→ Sufficient condition: the metric on $G \times M$ is invariant by the action

$$\left[\ \psi \cdot (\varphi, m) = (\varphi \circ \psi^{-1}, \psi \cdot m)\right.$$

Introduce notation:

$$\begin{cases} R_\varphi : \psi \longmapsto \psi\circ\varphi \\ R_m : \psi \longmapsto \psi\cdot m \\ A_\varphi : m \longmapsto \varphi\cdot m \end{cases}$$

Vertical space at (φ, m):

$$\left[\, V_{\varphi,m} = \left\{ (u,h) : dR_m(\varphi)u + dA_\varphi(m)h = 0 \right\} \right.$$

→ Indeed: The fiber above $m \in M$ is

$$\left[\pi^{-1}(m) = \left\{ \psi \cdot (id, m), \ \psi \in G \right\}\right.$$

→ by assumption: $T_{(\psi^{-1}, \psi \cdot m)}(G \times M) \underset{\text{isometric}}{\sim} T_{(id, m)} G \times M$

⇝ This isometry maps vertical spaces onto vertical spaces:

$$V_{id, m} = \left\{ (v, h) : \ v \cdot m + h = 0 \right\}$$

(id, m)

$(\psi^{-1}, \psi\cdot m)$

Verticality : $h = -v\cdot m$

$dA_{\psi^{-1}}(\psi\cdot m)\,\tilde{h} = -\,dR_{\psi\cdot m}(\psi^{-1})\,\tilde{v}$

$$h = -v\cdot m \xrightarrow{\text{isometry}} \Big(\underbrace{dR_{\psi^{-1}}(id)\,v}_{\tilde{v}}\,,\ \underbrace{dA_{\psi}(m).\,h}_{\tilde{h}}\Big)$$

$$dA_{\psi}(m).\,h = -\,dA_{\psi}(m)\,dR_m(id)\,v$$

and

$$dR_{\psi\cdot m}(\psi^{-1})\,dR_{\psi^{-1}}(id)\,v = \overbrace{dA_{\psi^{-1}}(\psi\cdot m)\,dA_{\psi}(m)}^{=\,Id}\,dR_m(id)\,v$$

$$
\begin{tikzcd}
\mathrm{id} \arrow[r, "R_m"] \arrow[dr, "R_{\psi^{-1}}"'] & m \arrow[r, "A_\psi"] & \psi \cdot m \arrow[r, "A_{\psi^{-1}}"] & m \\
 & \psi^{-1} \arrow[urr, "R_{\psi^{-1}\cdot m}"'] & &
\end{tikzcd}
$$

→ If an isometry maps vertical spaces to vertical spaces it also maps horizontal spaces to horizontal ones.

→ To build an invariant metric, it suffices to specify inner-products on tangent spaces at elements (id, m), for $m \in \mathcal{M}$.

→ Given these, one sets, for $(\varphi, m) \in G \times \mathcal{M}$

$$\left[\|(u,h)\|_{(\varphi,m)} = \| (\underbrace{dR_{\varphi^{-1}}(\varphi)\, u}_{v \in T_{id}G \Rightarrow \forall}, dA_{\varphi}(m)\, h \|_{(id, \varphi \cdot m)} \right.$$

→ Geodesic energy on $G \times M$:

$$\left[\int_0^1 \left\| \left(v, dA_\varphi(m)\dot{m}\right) \right\|^2_{\varphi \cdot m} \, dt \right.$$

<u>with</u> $v = dR_{\varphi^{-1}}(\varphi) \cdot \dot{\varphi}$

<u>i.e.,</u> $\dot{\varphi} = v \circ \varphi$

for a path $(\varphi(t), m(t))$, $t \in [0,1]$.

→ **Geodesic distance between fibers** : given μ_0, μ_1 :

minimize $\displaystyle\int_0^1 \| (v, dA_\varphi(m)\dot\varphi) \|^2_{id, \varphi\cdot m} \, dt$

over all paths $(\varphi(t), m(t))$ such that

$$\left[\begin{array}{ll} \partial_t \varphi = v \circ \varphi , & \varphi(0)\cdot m(0) = \mu_0 \\ & \varphi(1)\cdot m(1) = \mu_1 \end{array}\right.$$

→ There is no loss of generality in assuming $\varphi(0) = id$.

→ Make the <u>change of variables</u> $\mu(t) = \varphi(t) . m(t)$

→ <u>Then</u> $\dot{\mu} = v . \mu + dA_\varphi \dot{m}$

→ <u>Equivalent (Eulerian) formulation</u> : minimize

$$\int_0^1 \|(v, \dot{\mu} - v . \mu)\|^2_{(id,\mu)} \, dt$$

with $\mu(0) = \mu_0$, $\mu(1) = \mu_1$. [φ has disappeared !]

Image metamorphosis

- We have $\varphi \cdot m = m \circ \varphi^{-1}$; $v \cdot m = -\nabla m^T v$.
- Take $\|(v,h)\|^2_{(id,m)} = \|v\|^2_V + \frac{1}{\sigma^2}\|h\|^2_{L^2}$.
- Eulerian form: minimize

$$\int_0^1 \|v(t)\|_V^2 \, dt + \frac{1}{\sigma^2}\int_0^1 \|\partial_t \mu + \nabla \mu^T v\|^2_{L^2} \, dt$$

subject to $\mu(0) = m_0$, $\mu(1) = m_1$.

- **Lagrangian form**: $A_\varphi(m) = m \circ \varphi^{-1} \Longrightarrow$

$$dA_\varphi(m).h = h \circ \varphi^{-1}$$

- The Lagrangian formulation is then: minimize

$$\int_0^1 \|v(t)\|_V^2 \, dt + \frac{1}{\sigma^2} \int_0^1 \| \partial_t m \circ \varphi^{-1} \|_{L^2}^2 \, dt$$

subject to:

$$\begin{cases} \partial_t \varphi = v \circ \varphi \\ m(0) = m_0 \\ m(1) = m_1 \circ \varphi(1) \end{cases}$$

- Using a change of variables: minimize

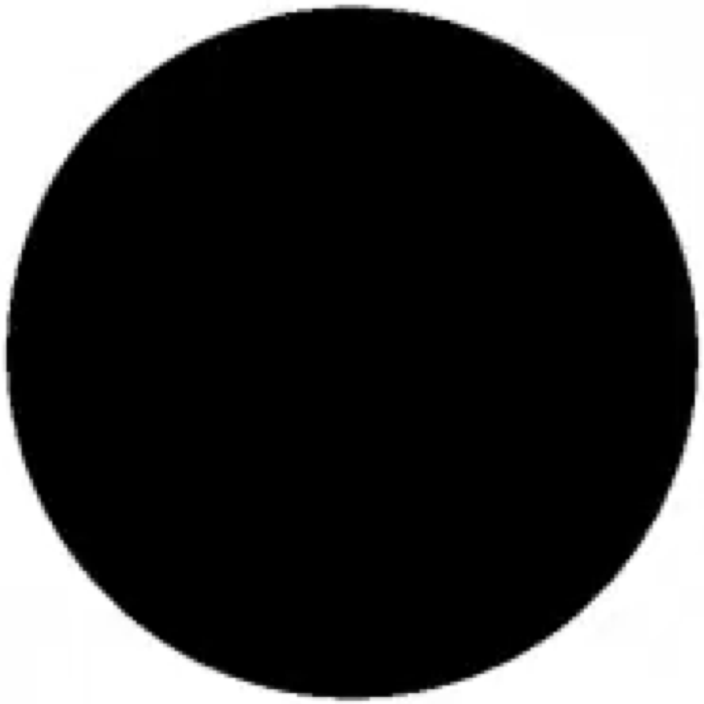
$$\int_0^1 \|v(t)\|_V^2 \, dt + \frac{1}{\sigma^2} \int_0^1 \int_{\mathbb{R}^d} \det(d\varphi(t,x)) \, |\partial_t m(t,x)|^2 \, dx \, dt$$

- For a fixed v, the optimal m can be computed analytically, yielding a reduced form minimizing

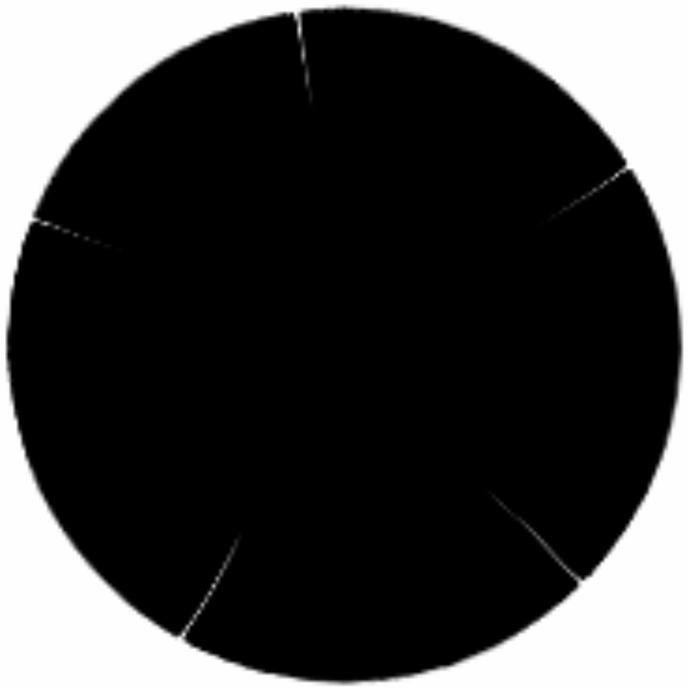
$$\int \|v(t)\|_V^2 \, dt + \frac{1}{\sigma^2} \int_{\mathbb{R}^d} C_t(\varphi) \, \|m_0 - m_1 \circ \varphi(1)\|^2 \, dt$$

with $C_t(\varphi) = [\det d\varphi(t)]^{-1} \Big/ \int_0^1 [\det d\varphi(s)]^{-1} \, ds$.

Metamorphosis



Deformed target.



Special case :

- Consider metamorphosis on functions $a : [0,1] \to S^{d-1}$
- Such functions can be associated with unit tangents of length-one curves expressed as functions of the arc length.

$\to$ The function $s \in [0,1] \longmapsto a(s) \in S^{d-1}$ represents the curve $m(s) = \int_0^s a(u)\,du$

- With previous notation:

$$\|v\|_V^2 = \int_0^1 |\partial_s v|^2 ds$$

for $v : [0,1] \to \mathbb{R}$, $v(0) = v(1) = 0$.

- Eulerian form: minimize

$$\int_0^1\int_0^1 |\partial_s v|^2 ds dt + \frac{1}{\sigma^2}\int_0^1\int_0^1 (\partial_t a + \partial_s a\, v)^2 ds dt$$

$$a(0,\cdot) = a_0 \; ; \; a(1,\cdot) = a_1$$

- Lagrangian form : replacing v by $\partial_t \varphi \circ \varphi^{-1}$

minimize

$$\int_0^1\int_0^1 \frac{(\partial_t \varphi)^2}{\partial_s \varphi}\, ds\, dt + \frac{1}{\sigma^2}\int_0^1\int_0^1 (\partial_t \alpha)^2\, \partial_s \varphi\, ds\, dt$$

$$\begin{cases} \alpha(0,\cdot) = a_0 \\ \alpha(1,\cdot) = a_1 \circ \varphi(1) \\ \varphi(0,\cdot) = id \end{cases}$$

Theorem $\sigma^2 \geq 1/2$

$\rightarrow$ if $a_0, a_1, \varphi(1) = \varphi_1$ are fixed, the metamorphosis minimal cost is equal to

$$4\arccos^2 \int_0^1 \sqrt{\varphi_1} \cos\left(\frac{\arccos(a_1 \circ \varphi_1(s)^T a_0(s))}{2\sigma}\right) ds$$

The squared metamorphosis distance between a_0 and a_1 is the minimum of this expression w.r.t. φ_1.

Optimal trajectories are deduced from geodesics on a unit sphere after a square root transformation.

All this is work a large group of people. Here are just a few:

U. Grenander; M. Miller; A. Trouvé; J. Glaunes; F.X. Vialard; T. Ratnanather; S. Arguillère; N. Charon; D. Tward; S. Joshi; M.F. Beg; M. Vaillant; A. Jain; D. Mumford; and many more...