Brief introduction to stochastic optimal control

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For stochastic optimal control problem is formulated with the features:

- the system state ↔ stochastic differential equations (SDE)
- control process ↔ stochatic process
- maximize (minimize) the objective functionals ↔ expection

Outline

I. introduce SDE
- Some elementary definitions / notions
  - probability space $(\Omega, \mathcal{F}, P)$
  - random variables
  - stochastic process $\{Y_s(\omega)\}_{s \geq t}$
  - filtration $\{\mathcal{F}_s\}_{s \geq t}$
  - control process $\alpha_s(\omega)$
- $\exists!$ strong solution of SDE

II. introduce value function

III. dynamic programming principle for Stochastic control

IV. the Hamilton-Jacobi-Bellman equation (HJBe)

I. Stochastic differential equation ↔ system state equation

I.1 Some basic definitions

Ω : sample space, the set of all possible results of random experiments

$\mathcal{F}$: σ-field, a family of some subsets of Ω and satisfies

- (1) $\Omega \in \mathcal{F}$;
- (2) if event $A \subset \Omega$, $A \in \mathcal{F}$, then $A^c = \Omega \setminus A \in \mathcal{F}$;
- (3) if $A_n \subset \Omega$, $n = 1, 2, \cdots$, $A_n \in \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$,

△ $\mathcal{F}$-measurable: If $A \subset \Omega$, $A \in \mathcal{F}$, then event A is called $\mathcal{F}$-measurable

$(\Omega, \mathcal{F})$: measurable space

P : the probability (measure) on $(\Omega, \mathcal{F})$, if $P(\cdot) : \mathcal{F} \to \mathbb{R}$ satisfies

- (1) $P(\Omega) = 1$
- (2) $\forall A \in \mathcal{F}$, $0 \leq P(A) \leq 1$
- (3) if $A_1, A_2, \cdots$ are mutually exclusive events, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

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$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

$\Rightarrow$

$(\Omega, \mathcal{F}, P)$: probability space

Random variable : Let $(\Omega, \mathcal{F}, P)$ be a probability space, the function $Y(\cdot): \Omega \to \mathbb{R}^d$. If for any $(y_1, y_2, \cdots y_d) \in \mathbb{R}^d$, the event
$$\{\omega : Y_1(\omega) \le y_1, \cdots, Y_d(\omega) \le y_d\} \in \mathcal{F},$$
then the $Y(\omega)$ is called random variable.

- domain : Ω
- "inverse" : $\mathcal{F}$ - measurable

Stochastic process : a family $(Y_s(\omega))_{s \in \mathcal{T}}$ of random variables defined on $(\Omega, \mathcal{F}, P)$.

- s : time parameter
  varing in $\mathcal{T}$ and maybe discrete or continuous
- $\mathcal{T}$: parameter space, $[t, T]$
  or $[t, \infty)$ for any $t \in [0, T]$.

Here, $\mathcal{F}$: the set of all possible events during $t \le s \le T$, includes past and future information.

But, in reality, at current time $s \in [t, T]$, we only know the past information (i.e., $t \le \tau < s$) and the information at time s; as time s envolves, one gets more and more information for some uncertain events.

$\therefore$ introduce a notation to describe the information known at time s and increases as time evolves.

Filtration on $(\Omega, \mathcal{F}, P)$:

A filtration on $(\Omega, \mathcal{F}, P)$ is an increasing family $\{\mathcal{F}_s\}_{s \ge t}$ of σ-fields of $\mathcal{F}$: $\mathcal{F}_\tau \subset \mathcal{F}_s \subset \mathcal{F}$ for all $t \le \tau \le s$.

Example: $\mathcal{F}_s^Y = \sigma(Y_\tau, t \le \tau \le s)$, $s \in [t, T]$

the smallest σ-field generated by Y_τ for all $t \le \tau \le s$ called the natural filtration of $\{Y_s\}_{s \ge t}$.

$(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s \ge t}, P)$: filtered probability space

A (stochastic) process is adapted :

A stochastic process $\{Y_s\}_{s \ge t}$ is adapted to the filtration $\{\mathcal{F}_s\}_{s \ge t}$, if for all $s \in [t, T]$, Y_s is $\mathcal{F}_s$ - measurable.

★ $\therefore$ an adapted stochastic process means that this process values at current time s only depends on the information known up to the present time s.

values at current time s only depends on the information known up to the present time s.
$\Downarrow$ $\mathcal{F}_s$.

Brownian motion

Definition 1.1.5 *(Standard Brownian motion)*
A standard d-dimensional Brownian motion on $\mathbb{T}$ is a continuous process valued in $\mathbb{R}^d$, $(W_t)_{t\in\mathbb{T}} = (W_t^1, \ldots, W_t^d)_{t\in\mathbb{T}}$ such that:

(i) $W_0 = 0$.

(ii) For all $0 \le s < t$ in $\mathbb{T}$, the increment $W_t - W_s$ is independent of $\sigma(W_u, u \le s)$ and follows a centered Gaussian distribution with variance-covariance matrix $(t-s)I_d$.

多元正态分布 $N(0, (t-s)I_d)$，均值为0.

Definition 1.1.6 *(Brownian motion with respect to a filtration)*
A vectorial (d-dimensional) Brownian motion on $\mathbb{T}$ with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ is a continuous $\mathbb{F}$-adapted process, valued in $\mathbb{R}^d$, $(W_t)_{t\in\mathbb{T}} = (W_t^1, \ldots, W_t^d)_{t\in\mathbb{T}}$ such that:

(i) $W_0 = 0$.

(ii) For all $0 \le s < t$ in $\mathbb{T}$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$ and follows a centered Gaussian distribution with variance-covariance matrix $(t-s)I_d$.

Of course, a standard Brownian motion is a Brownian motion with respect to its natural filtration.

$\Leftrightarrow$ (1) $P(W_0 = 0) = 1$; (2) $E_s[W_t - W_s] := E[W_t - W_s \mid \mathcal{F}_s] = 0$ $\quad 0 \le s < t$.
(3) $W_t - W_s \sim N(0, (t-s)I_d)$.

I.2 Stochastic differential equation

1. Consider the case — finite horizon : $[0,T]$, $T < +\infty$,

Fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\ge t}, P)$ satisfying usual conditions (i.e. $\{\mathcal{F}_s\}_{s\ge t}$ satisfies $\begin{cases}\text{right-continuous} \\ \text{complete}\end{cases}$ P2 of book written by Pham)

and

n-dimensional Brownian motion $B_s := (B^i)_{1\le i\le n}$ is adapted to $\{\mathcal{F}_s\}_{s\ge t}$.
$\Downarrow$ column vector

Assume that the system state $Y_s(\omega) : [t,T]\times\Omega \to \mathbb{R}^d$ is given by

$$\begin{cases} dY_s = f(Y_s, \alpha_s, s)ds + \sigma(Y_s, \alpha_s, s)dB_s, & s\in(t,T] \\ Y_t = x, \quad \text{a.s.} \end{cases} \tag{2.36}$$

$\begin{cases} q = l \text{ a.s.} : P(q = l) = 1. \\ \text{control process } \alpha_s := \alpha_s(\omega) : [t,T]\times\Omega \longrightarrow A \subset \mathbb{R}^m, \text{ measurable set} \\ \quad \text{and it is adapted to the filtration } \{\mathcal{F}_s\}_{s\ge t}. \\ \quad \to \text{column vector} \\ \text{vector } f = (f^i)_{1\le i\le d} : \mathbb{R}^d \times A \times [0,T] \longrightarrow \mathbb{R}^d, \text{ drift rate} \\ \text{matrix } \sigma = (\sigma^{ij})_{d\times n} : \mathbb{R}^d \times A \times [0,T] \longrightarrow \mathbb{R}^{d\times n}, \text{ diffusion rate} \\ (t,x) \in [0,T]\times\mathbb{R}^d, \text{ intial condition} \end{cases}$

$\therefore$ (2.36)

$$\Leftrightarrow \begin{cases} dY_s^i = f^i(Y_s, \alpha_s, s)ds + \sum_{j=1}^{n} \sigma^{ij}(Y_s, \alpha_s, s)dB_s^j \\ Y_t^i = x^i \end{cases}, \quad 1\le i\le d.$$

$$\Leftrightarrow \begin{cases} dY_s^i = f^i(Y_s, \alpha_s, s)\,ds + \sum_{j=1} \sigma^{ij}(Y_s, \alpha_s, s)\,dB_s^j \\ Y_t^i = x^i \end{cases}, \quad 1 \le i \le d.$$

2. Strong solution of SDE (∵ Brownian motion is not differentiable ∴ 对于 SDE 没有经典解的定义；另一种解的定义是弱解)

Def. (Strong solution) (§ 1.3 - book - pham , P22)

A strong solution of the SDE (2.36) Starting frome x at time t is a vectorial progressively measurable process $Y_s = (Y_s)_{1 \le i \le d}$ such that

i.e. for all $s \in [t,T]$, $(s,\omega) \mapsto Y_s(\omega)$ is $\mathcal{B}[t,s] \otimes \mathcal{F}_s$ measurable. P2 def. 1.1.3 book - pham.

$$\int_t^s |f(Y_\tau, \alpha_\tau, \tau)|\,d\tau + \int_t^s |\sigma(Y_\tau, \alpha_\tau, \tau)|^2 d\tau < \infty, \text{ a.s.}, \forall t \le s \le T$$

and

Itô integral (§ 1.2. book - pham)

$$Y_s = x + \int_t^s f(Y_\tau, \alpha_\tau, \tau)\,d\tau + \int_t^s \sigma(Y_\tau, \alpha_\tau, \tau)\,dW_\tau, \quad \text{a.s. } t \le s \le T$$

for all α_s and any intial condition $(t,x) \in [0,T] \times \mathbb{R}^d$.

Theorem (Existence and uniquness)

Assumptions:

(i) $\exists$ constants $C_1, C_2 > 0$ s.t. for all $t \in [0,T]$, $a \in A$, $y, z \in \mathbb{R}^d$

$|f(y,a,t) - f(z,a,t)| + |\sigma(y,a,t) - \sigma(z,a,t)| \le C_1 |y - z|$, (1) – lipschitz condition

(ii) Denote

$\mathcal{A}_{t,T} = \{\alpha_s(\omega) : [t,T] \times \Omega \to \mathbb{R}^m$ is adapted to the filtration $\{\mathcal{F}_s\}_{s \ge t}$ and satisfie (2) $\}$

$$E\left[\int_0^T |f(0,\alpha_s,s)|^2 + |\sigma(0,\alpha_s,s)|^2 ds\right] < +\infty \qquad (2)$$

Conclusions:

conditions (1) & (2) $\Rightarrow$ for all $\alpha_s \in \mathcal{A}_{t,T}$ and initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$, (2.36) $\exists!$ strong solution starting from x at time $s = t$, denoted by $\{Y_s^{t,x}\}_{s \ge t}$. Moreover,

$$E\left[\sup_{t \le s \le T} |Y_s^{t,x}|^2\right] < \infty \qquad (3)$$

and

$$\lim_{h \to 0^+} E\left[\sup_{s \in [t, t+h]} |Y_s^{t,x} - x|^2\right] = 0 \qquad (4)$$ (可以忽略不用)

as well as

$$Y_s^{t,x} = Y_s^{\theta, Y_\theta^{t,x}}, \quad \forall t \le \theta \le s, \quad t \in [0,T] \qquad (5)$$

will be used in proving dynamic programming principle

Proof of $\exists!$:

Step1. define a suitable Banach space E and the map

$$Y_s = x + \int_t^s f(Y_\tau, \alpha_\tau, \tau)\,d\tau + \int_t^s \sigma(Y_\tau, \alpha_\tau, \tau)\,d\tau$$

$$\hat{Y}_s = x + \int_t^s f(\hat{Y}_\tau, \alpha_\tau, \tau)\,d\tau + \int_t^s \sigma(\hat{Y}_\tau, \alpha_\tau, \tau)\,d\tau$$

$$Y_s = x + \int_t^s f(y_\tau, \alpha_\tau, \tau)d\tau + \int_t^s \sigma(y_\tau, \alpha_\tau, \tau)d\tau$$
$$\hat{Y}_s = x + \int_t^s f(\hat{y}_\tau, \alpha_\tau, \tau)d\tau + \int_t^s \sigma(\hat{y}_\tau, \alpha_\tau, \tau)d\tau$$
$\Rightarrow Y_s - \hat{Y}_s = \cdots$

Step2. By Burkholder-Gundy inequalities + estimates

$\Rightarrow$ map: $y(\cdot) \to Y(\cdot)$ from a suitable ball B in E to itself
the map is a contraction.

Step3. By Banach fixed-point theorem $\Rightarrow$ fixed point $\exists!$

Proof: (5)

$\because$ $\begin{cases} dY_s = f(Y_s, \alpha_s, s)dt + \sigma(Y_s, \alpha_s, s)dB_s, & s \geq t \\ Y_t = x \end{cases}$

$\exists!$ $Y_s^{t,x}$

$\begin{cases} dY_s = f(Y_s, \alpha_s, s)dt + \sigma(Y_s, \alpha_s, s)dB_s, & s \geq \theta \geq t \\ Y_\theta = Y_\theta^{t,x} \end{cases}$

$\exists!$ $Y_s^{\theta, Y_\theta^{t,x}}$

For $s \in [t, T]$

by pathwise uniqueness, $Y_s^{t,x} = Y_s^{\theta, Y_\theta^{t,x}}$. #

P23 + P38 << Continuous-time Stochastic Control and Optimization Financial Applications >> — Pham

II. Value function

非随机控制问题 is to find an optimal control α^* that maximize / minimize the

1. objective functional in the form:

$$J_{x,t}(\alpha(\cdot)) = E\left[\int_t^T r(Y_s, \alpha_s, s)ds + g(Y_T) \,\middle|\, Y_t = x\right]$$

$\begin{cases} \text{running payoff } r > 0 \\ \text{running cost } r < 0 \end{cases}$ $\begin{cases} \text{terminal payoff } g > 0 \\ \text{terminal cost } g < 0 \end{cases}$

expectation

for continuous random variable X:

$E[f(x)] = \int_{-\infty}^{\infty} f(x)\, g_X(x)\, dx$

g_X: probability density function

$= \int_{-\infty}^{\infty} f(x)\, d F_X(x)$

F_X: distribution function

$$= E\left[\int_t^T r(Y_s^{t,x}, \alpha_s, s)ds + g(Y_T^{t,x})\right].$$

Question: Is $J_{x,t}(\alpha(\cdot))$ well-defined?

To let $J_{x,t}(\alpha(\cdot))$ be well-defined, we can

2. assume that

(H1) $g(\cdot): \mathbb{R}^\alpha \to \mathbb{R}$ satisfies: $|g(y)| \leq C(1+|y|^2)$, $\forall y \in \mathbb{R}^\alpha$

this is not unique condition; C independent of y.

(H2) $r(y, a, s): \mathbb{R}^\alpha \times A \times [t, T] \to \mathbb{R}$ satisfies:

$|r(y, a, t)| \leq C(1+|y|^2) + \varkappa(a)$, $\forall (y, a, t) \in \mathbb{R}^\alpha \times A \times [0, T]$

(H2) $r(y, a, s): \mathbb{R}^d \times A \times [t,T] \to \mathbb{R}$ satisfies:

$$|r(y,a,t)| \le C(1+|y|^2) + \kappa(a), \quad \forall (y,a,t) \in \mathbb{R}^d \times A \times [0,T],$$

where C is a positive constant and $\kappa: A \to \mathbb{R}_+$ is a positive function satisfying

$$\kappa(a) \le C\left(1+|f(0,a,t)|^2 + |\sigma(0,a,t)|^2\right) \text{ for any } t \in [0,T],\ a \in A$$

$\Rightarrow$ for all control process $\alpha_s \in \mathcal{A}_{t,T}$.

$$E\left[\int_t^T |r(Y_s^{t,x}, \alpha_s, s)|\,ds\right] < \infty$$

$$E[|g(Y_s^{t,x})|] \le E[C + C|Y_s^{t,x}|^2] = C + C\,E[|Y_s^{t,x}|^2] \underset{\text{(3) property of strong solution}}{<} +\infty$$

$\Rightarrow J_{x,t}(\alpha(\cdot))$ is well-defined.

— P39 《Continuous-time Stochastic Control and Optimization Financial Applications》 — Pham

3. Then to maximize the expected payoff $\Rightarrow$

Value function:

$$u(x,t) = \max_{\alpha \in \mathcal{A}_{t,T}} E\left[\int_t^T r(Y_s^{t,x}, \alpha_s, s)\,ds + g(Y_T^{t,x})\right] \quad (2.39)$$

$\therefore$ for $\forall$ given $(t,x) \in [0,T] \times \mathbb{R}^d$, $\alpha^* \in \mathcal{A}_{t,T}$ is called optimal control if $u(x,t) = J_{x,t}(\alpha^*)$.

? If an optimal control does not exist, define the value function as (2.39)?

— 为啥可以? α^*不$\exists$, 那么 max 可能取不到吗, 咋定义成 (2.39) 呢? is more reasonable defined as sup?

— Yes

III. Dynamic Programming Principle for Stochastic Control

Theorem 2.3. *Let $u(x,t)$ be the value function defined by (2.39). If $t < \tau \le T$, then*

$$u(x,t) = \max_{\alpha \in \mathcal{A}_{t,\tau}} \mathbb{E}\left[\int_t^\tau r(Y_s, \alpha_s, s)\,ds + u(Y_\tau, \tau) \mid Y_t = x\right] \quad (2.41)$$

$$u(x,t) = \max_{\alpha \in \mathcal{A}_{t,\tau}} \mathbb{E}\left[\int_t^\tau r(Y_s, \alpha_s, s)\, ds + u(Y_\tau, \tau) \mid Y_t = x\right] \tag{2.41}$$

Proof: Exercise. The idea is the same as in the case of deterministic control. Split the integral into two pieces, one over $[t, \tau]$ and the other over $[\tau, T]$. Then condition on $\mathcal{F}_\tau$ and use the Markov property, so that the second integral and the payoff may be expressed in terms of $u(Y_\tau, \tau)$. □

Note. $\tau \in (t, T)$: stopping time

stopping time:

▲ Def. A random variable $\tau: \Omega \to (t, T)$ is a stopping time if for all $s \in (t, T)$

$$\{\underbrace{\tau \le s}_{\text{current time}}\} := \{\omega \in \Omega : \underbrace{\tau(\omega)}_{\text{the event } \omega \text{ first arrival time}} \le s\} \in \mathcal{F}_s.$$

▲

∴ Given the information in $\mathcal{F}_s$, $\{\tau \le s\} \in \mathcal{F}_s$ means the event ω happened before time s.

Proof. (2.39)

$$u(x,t) = \max_{\alpha \in A_{t,T}} E\left[\int_t^T r(Y_s^{t,x}, \alpha_s, s)\, ds + g(Y_T^{t,x})\right]$$

$$= \max_{\alpha \in A_{t,T}} E\left[\int_t^\tau r(Y_s^{t,x}, \alpha_s, s)\, ds + \int_\tau^T r(Y_s^{t,x}, \alpha_s, s)\, ds + g(Y_T^{t,x})\right]$$

property of strong solution: $Y_s^{t,x} = Y_s^{\tau, Y_\tau^{t,x}}$, $\forall\, t \le \tau \le s,\ t \in [0,T]$.

$$= \max_{\alpha \in A_{t,T}} E\left[\int_t^\tau r(Y_s^{t,x}, \alpha_s, s)\, ds + \underbrace{\int_\tau^T r(Y_s^{\tau, Y_\tau^{t,x}}, \alpha_s, s)\, ds + g(Y_T^{\tau, Y_\tau^{t,x}})}_{= J_{Y_\tau^{t,x}, \tau}(\alpha(\cdot))}\right]$$

$$= \max_{\alpha^{(1)} \in A_{t,\tau}} \max_{\alpha^{(2)} \in A_{\tau,T}} E\left[\int_t^\tau r(Y_s^{t,x}, \alpha_s^{(1)}, s)\, ds + J_{Y_\tau^{t,x}, \tau}(\alpha^{(2)}(\cdot))\right]$$

$$= \max_{\alpha^{(1)} \in A_{t,\tau}} E\left[\int_t^\tau r(Y_s^{t,x}, \alpha_s^{(1)}, s)\, ds + u(Y_\tau^{t,x}, \tau)\right]$$

$$= \max_{\alpha \in A_{t,\tau}} E\left[\int_t^\tau r(Y_s, \alpha_s, s)\, ds + u(Y_\tau, \tau) \mid Y_t = x\right].\ \#.$$

IV: The Hamilton-Jacobi-Bellman equation (HJBe)

1. Itô's formula:

Recall the chain rule

1. Itô's formula:

Recall the chain rule

$$\frac{dY(s)}{ds} = b(s), \quad s \in [t,T], \quad t \in [0,T]$$

for some $b(\cdot) \in L^1(t,T;\mathbb{R}^n)$. Then for any $F \in C^1([t,T]\times\mathbb{R}^d)$

$$d[F(s,Y(s))] = [F_s(s,Y(s)) + F_y(s,Y(s))b(s)]\,ds, \quad \forall s\in[t,T],\ t\in[0,T]$$

Consider Itô process

$$dY_s = f(Y_s,\alpha_s,s)\,dt + \sigma(Y_s,\alpha_s,s)\,dB_s, \quad s\in[t,T],\ t\in[0,T]$$

and let

$F(y,s) = (F^i(y,s))_{1\le i\le d} : [t,T]\times\mathbb{R}^d \to \mathbb{R}$ be C^1 in t and C^2 in y

f, σ satisfies some regularity, by formal calculations

$\Longrightarrow$ formally derive, strict proof see thm 4.1.2 - P44-46 - book - Bernt Øksendal, thm 4.2.1

$$\bullet\ \underset{1\times1}{dF(Y_s,s)} = \underset{1\times d}{\nabla F(Y_s,s)}\cdot \underset{d\times1}{dY_s} + F_s(Y_s,s)\,ds$$

$$+\ \frac{1}{2}(dY_s)^T_{1\times d}\,\underbrace{(D^2F(Y_s,s))_{d\times d}}_{\text{Hessian matrix}}\cdot dY_{s\ d\times1} + \underbrace{o(ds)}_{\text{剩下的项都是}\Delta s\text{的高阶无穷小量.}}$$

Hessian matrix
$$\begin{pmatrix} F_{y_1y_1} & \cdots & F_{y_1y_d} \\ \vdots & & \\ F_{y_dy_1} & \cdots & F_{y_dy_d}\end{pmatrix}$$

$$= \nabla F(Y_s,s)\cdot dY_s + F_s(Y_s,s)\,ds + \frac{1}{2}\sum_{i,j=1}^{d} dY_s^{(i)}\cdot dY_s^{(j)}\cdot F_{y_iy_j}$$

将向量相乘展开:

$$(dY^1,\dots,dY^d)_{1\times d}\begin{pmatrix} F_{y_1y_1} & \cdots & F_{y_1y_d} \\ \vdots & & \\ F_{y_dy_1} & \cdots & F_{y_dy_d}\end{pmatrix}_{d\times d}\cdot\begin{pmatrix} dY^1\\ dY^2\\ \vdots\\ dY^d\end{pmatrix}$$

$$= \left(\sum_{i=1}^{d} dY^i\cdot F_{y_iy_1},\ \dots\ \sum_{i=1}^{d} dY^i\cdot F_{y_iy_d}\right)\cdot\begin{pmatrix} dY^1\\ dY^2\\ \vdots\\ dY^d\end{pmatrix}$$

$$= \sum_{i=1}^{d} dY^i\cdot F_{y_iy_1}\cdot dY^1 + \sum_{i=1}^{d} dY^i\cdot F_{y_iy_2}\cdot dY^2 + \dots \sum_{i=1}^{d} dY^i F_{y_iy_d}\cdot dY^d.$$

$$= \sum_{i,j=1}^{d} dY^i\cdot dY^j\cdot F_{y_iy_j}$$

$$\because \begin{cases} dY_s^{(i)} = f^{(i)}(Y_s,\alpha_s,s)\,ds + \sum_{k=1}^{n}\sigma^{ik}(Y_s,\alpha_s,s)\,dB_s^{(k)}, & 1\le i\le d \\ dY_s^{(j)} = f^{(j)}ds + \sum_{k=1}^{n}\sigma^{jk}(Y_s,\alpha_s,s)\,dB_s^{(k)}, & 1\le j\le d\end{cases}$$

$$\therefore\ dY_s^{(i)}\cdot dY_s^{(j)} = f^{(i)}\cdot f^{(j)}(ds)^2 + f^i\,ds\cdot\sum_{k=1}^{n}\sigma^{jk}\cdot dB_s^{(k)}$$
$$+\ f^j\cdot ds\cdot\sum_{k=1}^{n}\sigma^{ik}\,dB_s^{(k)}$$

$$\therefore\ dY_s \cdot dY_s = f \cdot f^{j} (ds) + f\, ds \cdot \sum_{k=1} \sigma \cdots dB_s$$

$$+ f^{j} \cdot ds \cdot \sum_{k=1}^{n} \sigma^{ik}\, dB_s^{(k)}$$

$$+ \left(\sum_{k=1}^{n} \sigma^{ik}\, dB_s^{(k)} \right) \cdot \left(\sum_{k=1}^{n} \sigma^{jk}\, dB_s^{(k)} \right)$$

$$= f^{(i)} f^{(j)} (ds)^2 + f^{i} ds \cdot \sum_{k=1}^{n} \sigma^{jk} \cdot dB_s^{(k)}$$

$$+ f^{j} \cdot ds \cdot \sum_{k=1}^{n} \sigma^{ik}\, dB_s^{(k)}$$

$$+ \sum_{k=1}^{n} \sum_{k=1}^{n} \sigma^{ik} \sigma^{jk} \underbrace{dB_s^{(k)}\, dB_s^{(k)}}_{ds} \quad (b)$$

又 ∵ Brownian motion B_s satisfies

$$B_s - B_t \sim N(0, (t-s) I_d)$$

$\therefore$ for any $1 \leq i \leq d$, $B_s^{(i)} - B_t^{(i)} \sim N(0, t-s)$

$\therefore$ $Var[B_s^{(i)} - B_t^{(i)}] = E[(B_s^{(i)} - B_t^{(i)})^2] - E[B_s^{(i)} - B_t^{(i)}]^2$

$\Rightarrow$ $t-s = E[(B_s^{(i)} - B_t^{(i)})^2] - 0$

$\Rightarrow$ $ds := (dB_s^{(i)})^2$ } 形式上记作 ds, $\sqrt{ds}$.

$\Rightarrow$ $\sqrt{ds} := dB_s^{(i)}$

$\therefore$ the purple terms are higher-order infinitesimals, denoted as $o(ds)$

And the covariance

$$Cov(dB_s^i, dB_s^j) = E[(dB_s^i - E[dB_s^i])(dB_s^j - E[dB_s^j])]$$

$B_s^i - B_t^i \sim N(0, s-t)$

$$= \underbrace{E[dB_s^i \cdot dB_s^j]}_{\text{expectation}}$$

dB_s^i, dB_s^j independent random variable

$$\overset{i\neq j}{=} E[dB_s^i] E[dB_s^j]$$

$$= 0$$ 或者直接用"独立随机变量的协方差等于0)

$\therefore$

$$dY_s^{(i)} dY_s^{(j)} \overset{(b)}{=} \sum_{k=1}^{n} \sigma^{ik} \sigma^{jk} ds + o(ds)$$

$\Rightarrow$

$$dF(Y_s, s) = F_s(Y_s, s) ds + \underbrace{\nabla F(Y_s, s)}_{(F_{y_1}, \cdots F_{y_d})_{1\times d}} \cdot \overbrace{dY_s}^{f ds + \sigma dB_s}$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \sum_{i,j=1}^{d} \sigma^{ik} \sigma^{jk} F_{y_i y_j} + o(ds)$$

$\because dY_s = f ds + \sigma dB_s$ $\quad \nabla F = (F_{y_1}, \cdots, F_{y_d}) \cdot \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^d \end{pmatrix}$ $\quad d\times n$ matrix

$$= F_s(Y_s, s) ds + \nabla F(Y_s, s) \cdot f(Y_s, \alpha_s, s) ds + \nabla F(Y_s, s) \cdot \sigma(Y_s, \alpha_s, s) dB_s$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \sum_{i,j=1}^{d} \sigma^{ik} \sigma^{jk} F_{y_i y_j}(Y_s, s) ds + o(ds)$$

$n\times 1$ colum vector

$1\times d$ colum vector

$\therefore$

↙ Vector

∴ $1\times d$ colum vector

It$\hat{o}$'s formula: Consider It$\hat{o}$'s process

$$dY_s = f(Y_s, \alpha_s, s)ds + \sigma(Y_s, \alpha_s, s)dB_s,$$

for $F(y,s) \in C^{2,1}(\mathbb{R}^d, [0,T])$, one has

$\Rightarrow$

$$dF(Y_s,s) = F_s(Y_s,s)ds + [\nabla F(Y_s,s)\cdot f(Y_s,\alpha_s,s) + \frac{1}{2}\sum_{k=1}^{n}\sum_{i,j=1}^{d}\sigma^{ik}\sigma^{jk}F_{y_iy_j}(Y_s,s)]ds$$
$$+ \nabla F(Y_s,s)\cdot\sigma(Y_s,\alpha_s,s)dB_s$$
$$\overset{(☆1)}{=} F_s(Y_s,s)ds + [\nabla F(Y_s,s)\cdot f(Y_s,\alpha_s,s) + \frac{1}{2}tr(\sigma^T D^2F\sigma)]ds$$

$(n\times d)\times(d\times d)\times(d\times n) \rightarrow n\times n$ matrix

$$\overset{(☆2)}{=} F_s(Y_s,s)ds + \nabla F(Y_s,s)\cdot\sigma(Y_s,\alpha_s,s)dB_s$$
$$+ [\nabla F(Y_s,s)\cdot f(Y_s,\alpha_s,s) + \frac{1}{2}tr(D^2F(Y_s,s)\sigma(Y_s,\alpha_s,s)\sigma^T(Y_s,\alpha_s,s))]ds$$

$(dY_s)^2 \sim ds$, not $o(ds)$

~<< Stochastic Differential Equations >> - Bernt Øksendal, P44-46.

Proof of (☆1) & (☆2) 如下:

$\vee$ ∵ $\sigma^T D^2F\cdot\sigma$

$$= \underbrace{\begin{pmatrix} \hat{\sigma}^{11} & \hat{\sigma}^{12} & \cdots & \hat{\sigma}^{1d} \\ \vdots & & & \\ \hat{\sigma}^{n1} & \hat{\sigma}^{n2} & \cdots & \hat{\sigma}^{nd}\end{pmatrix}_{n\times d}}_{\sigma^T} \cdot \begin{pmatrix} F_{y_1y_1} & F_{y_1y_2} & \cdots & F_{y_1y_d} \\ \vdots & & & \\ F_{y_dy_1} & F_{y_dy_2} & \cdots & F_{y_dy_d}\end{pmatrix}_{d\times d} \cdot \underbrace{\begin{pmatrix} \sigma^{11} & \sigma^{12} & \cdots & \sigma^{1n} \\ \vdots & & & \\ \sigma^{d1} & \sigma^{d2} & \cdots & \sigma^{dn}\end{pmatrix}_{d\times n}}_{\sigma}$$

$$= \begin{pmatrix} \sum_{i=1}^{d}\hat{\sigma}^{1i}F_{y_iy_1} & \sum_{i=1}^{d}\hat{\sigma}^{1i}F_{y_iy_2} & \cdots & \sum_{i=1}^{d}\hat{\sigma}^{1i}F_{y_iy_d} \\ \vdots & & & \\ \sum_{i=1}^{d}\hat{\sigma}^{ni}F_{y_iy_1} & \sum_{i=1}^{d}\hat{\sigma}^{ni}F_{y_iy_2} & \cdots & \sum_{i=1}^{d}\hat{\sigma}^{ni}F_{y_iy_d}\end{pmatrix}_{n\times d} \begin{pmatrix} \sigma^{11} & \sigma^{12} & \cdots & \sigma^{1n} \\ & & & \\ \sigma^{d1} & \sigma^{d2} & \cdots & \sigma^{dn}\end{pmatrix}_{d\times n}$$

第一行第一列: $\sum_{i=1}^{d}\hat{\sigma}^{1i}F_{y_iy_1}\cdot\sigma^{11} + \sum_{i=1}^{d}\hat{\sigma}^{1i}F_{y_iy_2}\cdot\sigma^{21} + \cdots + \sum_{i=1}^{d}\hat{\sigma}^{1i}F_{y_iy_d}\cdot\sigma^{d1}$

$= \sum_{i,j=1}^{d}\hat{\sigma}^{1i}\sigma^{j1}F_{y_iy_j}$

第二行第二列: $\sum_{i=1}^{d}\hat{\sigma}^{2i}F_{y_iy_1}\cdot\sigma^{12} + \sum_{i=1}^{d}\hat{\sigma}^{2i}F_{y_iy_2}\cdot\sigma^{22} + \cdots + \sum_{i=1}^{d}\hat{\sigma}^{2i}F_{y_iy_d}\cdot\sigma^{d2}$

$= \sum_{i,j=1}^{d}\hat{\sigma}^{2i}\cdot\sigma^{j2}F_{y_iy_j}$

...

第n行第n列 $= \sum_{i,j=1}^{d}\hat{\sigma}^{ni}\sigma^{jn}F_{y_iy_j}$

$\Rightarrow tr[\sigma^T D^2F\sigma] = \sum_{k=1}^{n}\sum_{i,j=1}^{d}\hat{\sigma}^{ki}\sigma^{jk}F_{y_iy_j}$

$= tr[D^2F\sigma\cdot\sigma^T]$

∵ $\hat{\sigma}^{ki} = \sigma^{ik}$

$= \sum_{k=1}^{n}\sum_{i,j=1}^{d}\sigma^{ik}\sigma^{jk}F_{y_iy_j}$ #

$$= \sum_{k=1}^{n} \sum_{j=1}^{d} \alpha^{ik} \alpha^{jk} F_{y_i y_j} \quad \#$$

IV.2 HJBe

Step 1.

To use Itô's formula to derive HJBe, assume that

$$u(x,t) \in C^{2,1}(\mathbb{R}^d \times [0,T]).$$

Itô's formula $\Rightarrow$

$(u_{y_1}, u_{y_2}, \cdots, u_{y_d})_{1\times d} \cdot (\ )_{d\times n}$

$$u(Y_\tau, \tau) - u(x,t) = \int_t^\tau u_t(Y_s, s) + \mathcal{L}^\alpha u(Y_s, s)\, ds + \underbrace{\int_t^\tau \nabla u(Y_s,s)\, \sigma(Y_s, \alpha_s, s)\, dB_s}_{= J} \quad (2.42)$$

the term J is a martingale and $E[J]=0$

证明:

C1. **Definition 3.2.2** *A filtration (on $(\Omega, \mathcal{F})$) is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t\geq 0}$ of σ-algebras $\mathcal{M}_t \subset \mathcal{F}$ such that*

$$0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t$$

(i.e. $\{\mathcal{M}_t\}$ is increasing). *An n-dimensional stochastic process $\{M_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called a* martingale *with respect to a filtration $\{\mathcal{M}_t\}_{t\geq 0}$ (and with respect to P) if*

(i) M_t *is $\mathcal{M}_t$-measurable for all t,*
(ii) $E[|M_t|] < \infty$ *for all t*
and
(iii) $E[M_s|\mathcal{M}_t] = M_t$ *for all $s \geq t$.*

现在

future time 现在和过去的信息.

可以说是,若想在t时刻预测未来时刻s的value(资产?),会发现未来时刻s的资产期望和现在一样.∴无投机可言.

— << Stochastic Differential Equations >> — Bernt Øksendal

e.g. Brownian motion $(B_s)_{s\geq t}$ $\quad s \geq t$

$$
\begin{aligned}
& E[B_s | B_t] \\
&= E[(B_s - B_t) + B_t | B_t] \\
&= E[(B_s - B_t) | B_t] + E[B_t | B_t] \\
&= 0 + B_t \\
&= B_t
\end{aligned}
$$

C2. By corollary 3.2.6 (P33 Bernt) $\Rightarrow$

the Itô's integral

$$\int_t^\tau \nabla u(Y_s,s)\, \sigma(Y_s, \alpha_s, s)\, dB_s,$$

formally derive

$$=: \int_t^\tau h(Y_s, \alpha_s, s)\, dB_s =: M_\tau, \ \forall\ \tau \in [t,T],\ t \in [0,T]$$

$$=: \int_t^\tau h(Y_s, \alpha_s, s)\, dB_s =: M_\tau, \quad \forall\, \tau \in [t,T],\ t \in [0,T]$$

is a martingale w.r.t. the filtration $\{\mathcal{F}_\tau\}_{\tau \ge t}$.

$\to$ or say adapted

$$\Rightarrow \quad E[M_\tau \mid \mathcal{F}_t] = M_t = 0$$

By Law of Total Expectation about σ-field G

$$E[X] = E[\,E[X \mid G]\,]$$

$$\Rightarrow E[M_\tau] = E[\,E[M_\tau \mid \mathcal{F}_t]\,] = 0 \quad \#.$$

ie. $E[J] = 0$.

Step 2. 利用动态规划原理和 martingale (鞅)的性质

又 By DPP

$$u(x,t) = \max_{\alpha \in A_{t,T}} E\left[\int_t^\tau r(Y_s, \alpha_s, s)\,ds + u(Y_\tau, \tau) \,\Big|\, Y_t = x\right]$$

$$\because \max_{\alpha \in A_{t,T}} E[u(x,t) \mid Y_t = x] = E[u(x,t) \mid Y_t = x] = u(x,t)$$

$\because$ initial condition (t,x), $u(x,t) \sim$ fixed value $\sim$ 'constant'

$\therefore$

$$0 = \max_{\alpha \in A_{t,T}} E\left[\int_t^\tau r(Y_s, \alpha_s, s)\,ds + u(Y_\tau, \tau) - u(x,t) \,\Big|\, Y_t = x\right]$$

$\therefore$ 将(2.42)代入上式并利用 $E\left[\int_t^\tau \nabla u\, \sigma\, dB_s\right] = 0 \Rightarrow$

$$0 = \max_{\alpha \in A_{t,\tau}} E\left[\int_t^\tau r(Y_s, \alpha_s, s)\,ds + \int_t^\tau u_t(Y_s, s) + \mathcal{L}^\alpha u(Y_s, s)\,ds \,\Big|\, Y_t = x\right] \qquad (2.44)$$

Step3. 取极限.

let $\tau = t + h, \quad h \to 0$

$$0 = \max_{a \in A} E\left[\, r(x,a,t) + u_t(x,t) + \mathcal{L}^\alpha u(x,t)\,\right]$$

$E[u_t(x,t)] = u_t(x,t)$. 因为对于任意固定的 initial condition (t,x), $u_t(t,x)$ 相当于是固定常数. $\therefore E[c] = c$.

$$\Longrightarrow \quad u_t(x,t) + \max_{a \in A} E\left[\, r(x,a,t) + \mathcal{L}^\alpha u(x,t)\,\right] = 0$$

$\Rightarrow$

$$u_t(x,t) + \max_{a \in A} E\Big[r(x,a,t) + \tfrac{1}{2}\operatorname{tr}\left(D^2 u(x,t)\,\sigma(x,a,t)\,\sigma^T(x,a,t)\right) + \nabla u(x,t) \cdot f(x,a,t) \Big] = 0 \qquad (2.46)$$

$$u_t(x,t) + \max_{a\in A} E[r(x,a,t) + \frac{1}{2} tr(\nabla^2 u(x,t)\sigma^T(x,a,t)\sigma(x,a,t)) + \underline{\nabla u(x,t)\cdot f(x,a,t)}] = 0 \qquad (2.46)$$

$$\begin{cases} \nabla u(x,t) = (\frac{\partial u}{\partial x_1}, \cdots \frac{\partial u}{\partial x_d}) \\ f = \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \end{cases}$$

$$\Rightarrow \quad u_t + H(D^2u, Du, x, t) = 0.$$

V. Infinite horizon problem; $T = +\infty$

key point: $\begin{cases} \sigma, f, r \text{ are independent of } t. \\ \text{no terminal payoff} \end{cases}$