2024年9月30日 21:13

Recall:

1. system state equation

$Y_s(\omega): [t,T]\times\Omega \to \mathbb{R}^d$

$$\begin{cases} dY_s = f(Y_s,\alpha_s,s)\,ds + \sigma(Y_s,\alpha_s,s)\,dB_s. \quad s\ge t \\ Y_t = x, \quad a.s. \end{cases} \qquad (2.36)$$

filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{s\ge t}, P)$.

2. the set of admissible control:

$\mathcal{A}_{t,T} = \{\alpha_s(\omega): [t,T]\times\Omega \to A\subset\mathbb{R}^m \mid \alpha_s$ is adapted to the filtration $\{\mathcal{F}_s\}_{s\ge t}\}$.

3. existence and uniqueness of strong solution (s.s.) to (2.36)

Supposing σ & f satify usual bounds + continuity condition, then for any given initial condition (x,t) and $\alpha_s(\omega)$, (2.36) $\exists!$ strong solution.

3. Value function & dynamic programming principle

Assumptions: $\begin{cases} g(\cdot) \\ r \end{cases}$ satisfies $\left.\begin{matrix}(H_1)\\(H_2)\end{matrix}\right\}$ given in last Tuesday

$\Rightarrow$

- objective function

$$J_{x,t}(\alpha(\cdot)) = E\left[\int_t^T r(Y_s,\alpha_s,s)\,ds + g(Y_T) \,\middle|\, Y_t = x\right] < +\infty$$

☆ ocp is to maxize (or minimize) the objective function over all admissible controls

$\Rightarrow$

- Value function

$$u(x,t) = \max_{\alpha\in\mathcal{A}_{t,T}} E\Big[\underbrace{\int_t^T r(Y_s,\alpha_s,s)\,ds}_{\text{running payoff}} + \underbrace{g(Y_T)}_{\text{terminal payoff}} \,\Big|\, Y_t = x\Big\} \qquad (2.39)$$

- DPP

for any stopping time $\tau\in(t,T]$,

$$u(x,t) = \max_{\alpha\in\mathcal{A}_{t,\tau}} E\left[\int_t^\tau r(Y_s,\alpha_s,s)\,ds + u(Y_\tau,\tau) \,\middle|\, Y_t = x\right\} \qquad (2.41)$$

4. The Hamilton-Jacobi-Bellman equation (HJBe)

Assume $u(x,t)\in C^{2,1}(\mathbb{R}^d\times[0,T])$. formally derive

$$u_t(x,t) + \max_{a\in A}\Big[r(x,a,t) + \underbrace{\tfrac12 \mathrm{tr}(D^2u(x,t)\sigma(x,a,t)\sigma^T(x,a,t)) + f(x,a,t)\cdot\nabla u(x,t)}_{= \mathcal{L}^a u}\Big] = 0$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\shortparallel} \quad \underbrace{\qquad\qquad}_{\mathcal{L}^{\alpha}u}$$

$$H(D^2u, Du, x, t)$$ the second order differential operator

$\Leftrightarrow$

$$u_t(x,t) + H(D^2u, Du, x, t) = 0 \qquad (2.47)$$

总结:

Why need HJBe : find an optimal control α^* s.t. $u(x,t) = J_{x,t}(\alpha^*)$.

有两种框架去find optimal control:

☆ way 1: in classical PDE approach: here a prior assumption: $u(x,t) \in C^{2,1}$

way 2: in viscosity solution approach; weaker assumption: $u(x,t)$ locally bdd

Today,

I. The classical PDE approach

(that is the)

Verification theorem (see P47 -book- pham)

Assumptions:

① $w(x,t) \in C^{2,1}(\mathbb{R}^d \times [0,T)) \cap C^0(\mathbb{R}^d \times [0,T])$

② $w(x,t)$: $\exists$ a constant $C>0$ s.t. growth condition

$|w(x,t)| \le C(1+|x|^2)$, $\forall (x,t) \in [0,T] \times \mathbb{R}^d$

③ $w(x,T) = g(x)$ terminal payoff

④ $\exists$ a measurable function $\hat{\alpha}(x,t): \mathbb{R}^d \times [0,T) \to A \subset \mathbb{R}^m$

s.t.

$$w_t + \max_{a\in A}[r(x,a,t) + \mathcal{L}^a w(x,t)] = 0, \text{and} \max_{a\in A}[r(x,a,t) + \mathcal{L}^a w(x,t)] = r(x,\hat{a},t) + \mathcal{L}^{\hat{a}} w(x,t)$$

(i.e, $\exists$ a function $\hat{\alpha}(x,t)$, s.t. $w(x,t)$ satisfies HJBe)

⑤ The SDE
$$\begin{cases} dY_s = f(Y_s, \hat{\alpha}_s, s)ds + \sigma(Y_s, \hat{\alpha}_s, s)dB_s \\ Y_t = x \end{cases}$$

$\exists!$ Solution, denoted by $\hat{Y}_s^{t,x}$;

And the control $\{\hat{\alpha}(\hat{Y}_s^{t,x}, s)\}_{s\ge t} \in \mathcal{A}_{t,T}$.

Conclusion:

① $J_{x,t}(\hat{\alpha}) = w(x,t) = u(x,t)$

② $\therefore \hat{\alpha}$ is an optimal control.

总结: the classical approach, 就是

HJBe 的解不一定有这么好的正则性. ∴ 引入粘性解的框架.

Step1. obtain the existence of smooth solution to HJBe for some $\hat{\alpha}$

Step2. By verification thm, show this smooth solution is the value function

Step3. obtain the optimal control $\hat{\alpha}$

step3. obtain the optimal control $\hat{\alpha}$

II. Example:

1. System state equation

Consider a wealth (bond & stock) process evolves according to

$$\begin{cases} dX_s = X_s[\underbrace{\alpha_s\mu + (1-\alpha_s)r}_{f}]ds + X_s\alpha_s\beta dB_s, & s \geq t,\ t \in [0,T] \\ X_t = x \end{cases}$$

SDE

$(1-\alpha_s)$: the bond proportion

α_s: the stock proportion of wealth

μ, r, σ - constants

2. admissible control set → 投资组合约束限制

The investor faces the portfolio constraint, i.e.

$$A := \left\{ \begin{array}{l} \alpha_s: 1)\ \alpha_s: [0,T]\times\Omega \longrightarrow A \subset \mathbb{R},\ A \text{ is a closed convex subset} \\ 2)\ \alpha_s \text{ is progressively measurable, s.t. } \int_0^T |\alpha_s|^2 ds < \infty \end{array} \right\}$$

→ integrability

i.e. for all $s \in [t,T]$, α_s is $\mathcal{B} \otimes \mathcal{F}_s$ measurable.

3. objective function:

the expected utility from terminal wealth at T

$$J_{x,t}(\alpha(\cdot)) = E[g(X_T^{t,x})],\ (t,x) \in [0,T]\times\mathbb{R}^+$$

with

$$g(x) = \frac{x^p}{p},\ x \geq 0,\ p<1,\ p \neq 0.$$

∴ running payoff $r \equiv 0$.

4 Value function

$$V(x,t) = \max_{\alpha\in A} E[g(X_T^{t,x})] \qquad (1)$$

5. HJBe for the stochastic C.P. (1)

$$\begin{cases} W_t + \max\limits_{\alpha\in A}[r(x,a,t) + \frac{1}{2}tr(D^2W(x,t)\sigma(x,a,t)\sigma^T(x,a,t)) + f\cdot\nabla W] = 0 \\ W(x,T) = g(x), \quad x \in \mathbb{R}^+ \end{cases}$$

$\in \mathbb{R}^+$

1 dimensional

$\mathcal{L}^\alpha = b(x,t)\cdot\nabla W + \frac{1}{2}tr(D^2W\sigma\cdot\sigma^T)$

terminal condition

$\Rightarrow$

$b(x,t) =$

$$W_t + \max_{\alpha\in A}[x(a\mu + (1-a)r)W_x + \frac{1}{2}x^2a^2\beta^2 W_{xx}] = 0$$

!!

random walk

(2)

va...

$\mathcal{L}^a w(x,t)$ random work never reach boundary, so here don't need the boundary condition (2)

$w(x,T) = g(x) = \frac{x^p}{p} > 0$, $x \in \mathbb{R}^+$, $\boxed{p<1, p\neq 0}$ $\;$ $p=0$, $g(x) = \lim_{p\to 0}\left(\frac{x^p}{p} - \frac{1}{p}\right)$

为了通过 HJBe (2) 找到最优 control α,

Step 1. find a smooth solution of (2) for some $\hat{\alpha}$

1) assume the smooth solution in the form

$$\boxed{w(x,t) = \phi(t) g(x) \quad \text{for some } \phi(t) > 0.}$$

$g(x) = \frac{x^p}{p}$

$g_{xx}(x) = p \cdot (p-1) x^{p-2}$

$\Rightarrow w_t = \phi_t \, g(x)$

$$\begin{aligned}
\mathcal{L}^a w(x,t) &= x(a\mu + (1-a)r)\phi(t) g_x(x) + \tfrac{1}{2} x^2 a^2 \beta^2 \phi(t) g_{xx}(x) \\
&= x\, x^{p-1}[a(\mu - r) + r]\phi(t) + \tfrac{1}{2} a^2 \beta^2 \phi(t) \cdot x^2 \cdot (p-1) x^{p-2} \\
&= p\, g(x) [a(\mu-r) + r]\phi(t) + \tfrac{1}{2}(p-1) a^2 \beta^2 \phi(t) \cdot p\, g(x) \\
&= p\phi(t) g(x) [a(\mu - r) + r - \tfrac{1}{2}(1-p) a^2 \beta^2] \\
&=: p\phi(t) g(x) F(a).
\end{aligned}$$

$w(x,T) = \phi(T) g(x) = g(x)$

$$\Leftrightarrow \begin{cases} \phi_t g(x) + p \max\limits_{a\in A}[F(a)\phi(t) g(x)] = 0 \\ \phi(T) g(x) = g(x) \end{cases} \quad (3)$$

$= \max\limits_{a\in A} \mathcal{L}^a(\phi(t) g(x))$.

with $F(a) = a(\mu - r) + r - \frac{1}{2}(1-p) a^2 \beta^2$.

$g(x) > 0$ $\quad g(x) = \frac{x^p}{p} > 0$, terminal function

$\Leftrightarrow$

a new ODE:

$$\begin{cases} \phi_t + \underbrace{p\max\limits_{a\in A}[F(a)]}_{=:q} \phi(t) = 0 \\ \phi(T) = 1 \end{cases} \quad (4)$$

$$\Rightarrow \phi(t) = e^{q(T-t)}$$

$$\therefore w(x,t) = e^{q(T-t)} \cdot g(x), \quad (x,t) \in \mathbb{R}^+ \times [0,T]$$

$$= e^{q(T-t)} \cdot \frac{x^p}{p} \in C^{2,1}(\mathbb{R}^+ \times [0,T]),$$

with $q = p\max\limits_{a\in A}[F(a)]$.

$\Rightarrow w(x,t) = e^{q(T-t)} \frac{x^p}{p}$ is a smooth solution of HJBe (3):

$$\begin{cases} w_t + \boxed{\max\limits_{a\in A}[F(a) p w(x,t)]} = 0, \quad (x,t) \in \mathbb{R}^+ \times [0,T). \end{cases}$$

$$\begin{cases} w_t + \max_{a\in A}\left[F(a)\,p\,w(x,t)\right] = 0, & (x,t)\in \mathbb{R}^+\times[0,T). \\ \quad\quad\quad \max_{a\in A}\mathcal{L}^a w(x,t) & \\ w(x,T) = g(x) & x\in\mathbb{R}^+ \end{cases} \quad (5)$$

2) 实际上，对于这个 red term, $\exists\ \hat{a}\in A$, s.t. red term $= F(\hat{a})\,p\,w(x,t)$:

2) $\because F(a) = a(\mu-r) + r - \frac{1}{2}a^2(1-p)\beta^2$ $\quad g(x)=\frac{x^p}{p},\ p<1,$

$\Rightarrow F'(a) = \mu - r - a(1-p)\beta^2$

$F''(a) = -(1-p)\beta^2 < 0\ (\because p<1) \to$ concave function

and A is a closed convex set

$\therefore \exists!\ \hat{a}\in A$, s.t. $\max_{a\in A} F(a) = F(\hat{a})$.

$\because w(x,t) = \phi(t)g(x) > 0$

$$\therefore \max_{a\in A}\mathcal{L}^a(w(x,t)) = p\max_{a\in A}\left[F(a)\cdot\phi(t)g(x)\right]$$
$$= p\cdot F(\hat{a})\phi(t)g(x).$$

$\therefore$ for this $\hat{a}$, HJBe (3) $\exists$ a smooth solution

$$w(x,t) = e^{pF(\hat{a})(T-t)}\cdot\frac{x^p}{p},$$
$$\le e^{pF(\hat{a})T}\frac{1}{p}x^p, \quad p<1$$
$$\le \begin{cases} e^{pF(\hat{a})T}\frac{1}{p}x^2. & \text{for any } x\ge 1 \\ e^{pF(\hat{a})T}\frac{1}{p} + x^2 & 0<x<1 \end{cases}$$

here $w(x,t)$ satisfies growth condition.

Step 2. for this constant control $\hat{a}$, consider the system state equation

$$\begin{cases} dX_s = X_s[\hat{a}\mu + (1-\hat{a})r]\,ds + X_s\hat{a}\beta\,dB_s, & s\ge t,\ t\in[0,T] \\ X_t = x \end{cases}$$

here $\begin{cases} f(x,a) = x[a\mu + (1-a)r] \\ \sigma(x,a) = xa\beta \end{cases}$ satisfies Lipschitz and linear growth

$\Rightarrow$ the SDE $\exists!$ strong solution $\hat{X}_s^{t,x}$

&

$\hat{a}\in A$.

$$\xrightarrow[\text{theorem}]{\text{Verification}} \boxed{J_{x,t}(\hat{a}) = w(x,t) = u(x,t)}$$

$\Rightarrow$ the maximum point $a=\hat{a}\in A$ for $F(a) = a(\mu-r) + r - \frac{1}{2}a^2(1-p)\beta^2$.

$\Rightarrow$ the maximum point $\alpha=\hat{\alpha}\in A$ for $F(a) = a(\mu-r)+r-\frac{1}{2}a^2(1-p)\beta^2$, is the optimal control. #.

Remark.

1. To use verification theorem, the uniqueness of smooth solution to HJBe need to get.

$\therefore$ Generally, we need consider

$$\begin{cases} \text{HJBe} \\ \text{terminal condition} \\ \text{boundary condition} \end{cases}$$

but, here, there is no boundary condition in the example. This is because:

1) the second order coefficient of $\mathcal{L}^{\alpha} w := x(a\mu+(1-a)r)w_x + \frac{1}{2}x^2a^2\beta^2 \cdot w_{xx}$ is $\frac{1}{2}x^2a^2\beta^2$;

2) 而 random work never reach boundary.