The framewok of optimal switching problem

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finite horizon $\underset{\Downarrow}{T < +\infty}$ and $\underset{\Downarrow}{\text{bdd domain}}$.

terminal condition boundary condition

1. Background

- An oil company operates across three production lines:

$$\left.\begin{cases} \text{crude oil} & i=1 \\ \text{natural gas} & i=2 \\ \text{refined oil} & i=3 \end{cases}\right\} \text{ regime states}$$

The price X fulfills a SDE and its dynamics may differ according to the regime states;

并且 X can't increase/decrease to $+\infty/-\infty$, $0 \le X \le L$ → 某个常数.

The running payoff/cost depends on market price X.

- Objective:

Find the $\boxed{\text{strategy}}$ that maximizes the expected profits

e.g., at time $s=t$, producing crude oil (i.e., $i=1$).

A strategy: when switch $i=1$ & where switch to (2 or 3)?

2. Problem formulation

We formulate this problem into the framework ...

2.1 Setup and assumptions

- finite horizon: $t \in [0,T]$, $T < +\infty$ with bdd open set $\mathcal{O} \subset \mathbb{R}^d$, $d \ge 1$
- filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_s)_{t \le s \le T}, P)$
- define the set of regimes: $\mathbb{I}_m = \{1, 2, 3, \cdots, m\}$.
- switching control is a double sequence: $\alpha = (\tau_n, \iota_n)_{n \ge 1}$

  τ_n: increasing sequence of stopping time; ι_n: valued in $\mathbb{I}_m$ during $[\tau_n, \tau_{n+1})$

- $\mathcal{A}$: the set of switching controls.
- For a given initial regime state $i \in \mathbb{I}_m$ at t, $\alpha \in \mathcal{A}$, define a controlled switching process

$$\begin{cases} I_s^i = \sum_{n \ge 0} \iota_n \mathbb{1}_{[\tau_n, \tau_{n+1})}(s), \quad s \ge t, \\ I_{t^-}^i = i \end{cases}, \quad \text{if } s \in [\tau_n, \tau_{n+1}),\ I_s^i = \iota_n$$

with $\tau_0 = t$, $\iota_0 = i$, $t \in [0, T)$.

2.2 Diffusion system:

- The price X: $W_t := (B^i)_{1 \le i \le n}$ on $(\Omega, \mathcal{F}, \mathbb{F}, P)$

$$\begin{cases} dX_s^{t,x,i} = b(X_s^{t,x,i}, I_s^i)\,ds + \sigma(X_s^{t,x,i}, I_s^i)\,dW_t & (2.1) \\ (t,x,i)\text{: initial condition} \end{cases}$$

i: initial regime state at $s=t$. $\{ X_t = x$.

Denote $\begin{cases} b(x,i) =: b_i \in \mathbb{R}^d \\ \sigma(x,i) =: \sigma_i \in \mathbb{R}^{d \times n} \end{cases}$

By [Kharroubi-2016-SIAM or Protter-2005-book-Thm V.38 + Chapter 3 in Pham's book]

$[\sigma(x,i) =: \sigma_i \in \mathbb{R}$

By [Kharroubi-2016-SIAM or Protter-2005-book-Thm V.38 + Chapter 3 in Pham's book]

For given $(t,x,i) \in [0,T]\times O \times \mathbb{I}_m$, $O \subset \mathbb{R}^d$, open connected set

(2.1) $\exists!$ strong solution $\boxed{X_s^{t,x,i}}$, satisfying $\mathbb{E}[\sup_{t\le s\le T} |X_s^{t,x,i}|^2] < \infty$,

provided that

(H1) $|b(x,i)-b(y,i)| + |\sigma(x,i)-\sigma(y,i)| \le L|x-y|$ for some constant $L>0$, $x,y\in O$.

整个过程考虑以下特征:

- (Ha)

running payoff function:

$0 \le f(t,x,j) : [0,T]\times \bar{O} \times \mathbb{I}_m \to \mathbb{R}$, denoted by $f_j(t,x)$,

terminal payoff function

$0 \le \psi(t,x,j) : T \times \bar{O} \times \mathbb{I}_m \to \mathbb{R}$, denoted by $\psi_j(x)$ (\psi:)

$t=T$

Lipschitz continuous w.r.t. x

(Hb)

discount rate

$\gamma \le \beta(t,x,j) : [0,T]\times \bar{O} \times \mathbb{I}_m \to \mathbb{R}$; $\beta_j(t,x)$

for some positive constant γ.

If $\begin{cases} T=+\infty, & \gamma \text{ needs to be positive} \\ T<+\infty, & \gamma \ge 0. \end{cases}$

(Hc)

switching cost

$C_{ij}(t,x) : [0,T]\times \bar{O} \to \mathbb{R}$, $i,j \in \mathbb{I}_m$, the cost switch from i to j; and

- $0 \le C_{ij}(t,x) \in C^{1,2}([0,T]\times \bar{O})$, " = " ✓ $\Leftrightarrow i=j$;
- $C_{kj} < C_{kq} + C_{qj}$, $q \ne k, j$. — more cost, $k\to q\to j$ than $k\to j$ directly
- $\psi_i(x) \ge \max_{i\ne j} (\psi_j(x) - C_{ij}(T,x))$ — compatible with terminal payoff

Based on references: Fleming-2006-book-Remark 8.2, pp. 221, Lemma 7.1, pp. 218
Olofsson-2022-AMO
Hamadène-2023-JMAA

$\Rightarrow$ for a given control α:

2.3 Profit function:

$$J(t,x,i,\alpha) = \mathbb{E}\Big[\int_t^T e^{-\beta_{I_s^i}(s,X_s^{t,x,i})(s-t)} f_{I_s^i}(s,X_s^{t,x,i})\,ds \quad \text{running payoff}$$

$$\overset{!!}{=} J_i(t,x,\alpha) \qquad - \sum_{t\le \tau_n < T} e^{-\beta_{I_s^i}(s,X_s^{t,x,i})(\tau_n - t)} C_{l_{n-1},l_n}(s,X_s^{t,x,i}) \quad \text{switching cost}$$

$$+ e^{-\beta_{I_T^i}(T,X_T^{t,x,i})(T-t)} \psi_{I_T^i}(T,X_T^{t,x,i}) \quad \text{terminal payoff.}$$

$$+ e^{-\beta_{I_T^i}(T, X_T)(T-t)} \psi_{I_T^i}(T, X_T^{t,x,i}))$$ → terminal payoff.

$\beta_j, f_j, \psi_j, C_{l_{n-1}, l_n} \Rightarrow J_i(t,x,\alpha) < +\infty$, ~~its well-defined~~. [Chapter 3, Chapter 5 in Pham's book]

Objective:

Find α strategy to maximize $J_i(t,x,\alpha)$, i.e.,

Find α α^* s.t

$$J_i(t,x,\alpha^*) \geq J_i(t,x,\alpha) \text{ for all } \alpha \in A$$

(i.e., $$J_i(t,x,\alpha^*) = \sup_{\alpha\in A} J_i(t,x,\alpha) =: V_i(t,x). \quad (2.3)$$)

这里利用粘性解方法解决这个 optimal control problem.

3. Hamilton-Jacobi-Bellman equation

[Kharroubi-2016-SIAM, thm 5.1]

Lemma 3.1 (Stochastic Dynamic Programming Principle (SDPP))

For any $(t,x,i) \in [0,T] \times O \times \mathbb{I}_m$,

$$V_i(t,x) = \sup_{\alpha\in A_{t,\theta}} E\Big[\int_t^\theta e^{-\beta_{I_s^i}(s, X_s^{t,x,i})(s-t)} f_{I_s^i}(s, X_s^{t,x,i})\, ds$$

$$- \sum_{t\le \tau_n < \theta} e^{-\beta_{I_s^i}(s, X_s^{t,x,i})(\tau_n - t)} C_{l_{n-1}, l_n}(s, X_s^{t,x,i})$$

$$+ e^{-\beta_{I_\theta^i}(\theta, X_\theta^{t,x,i})(\theta-t)} V_i(\theta, X_\theta^{t,x,i})\Big)$$

$A = \{(\tau_n, l_n)_{n\ge1}, t \le \tau_n < T\}$

$A_{t,\theta} = \{(\tau_n, l_n)_{n\ge1}, t \le \tau_n \le \theta\}$

$\theta \in [t,T]$ is any stopping time.

首先，给出一个形式上的讨论: Formal discussion

类似之前讲的，形式上，若取 $\theta = t+h$, h small enough

① If no switch $(t, t+h)$

SDPP $\xrightarrow{\text{formally}}$ HJBe ? has introduced in stochastic optimal control.

$$-\partial_t V_i + \beta_i V_i - \mathcal{L}_i V_i - f_i = 0 \quad \leftarrow \text{formal}$$

with $\mathcal{L}_i = b_i \cdot D_x\varphi + \frac{1}{2}\mathrm{tr}(\sigma_i \sigma_i'(x) D_x^2 \varphi)$ — generator of the diffusion X in regime i

$(t, t+h)$, no control $\Rightarrow$ no sup.

② 另一方面, switch from i to j, switching cost

1) payoff $= V_j(t,x) - C_{ij}$

2) maximize

$$\max_{j\ne i} (V_j(t,x) - C_{ij}) = V_i(t,x) \quad \to \text{optimal payoff}$$

i.e., optimal payoff $$V_i(t,x) - \max_{j\ne i}(V_j(t,x) - C_{ij}) = 0$$

⇒ (two formulas)

$$\min\{-\partial_t V_i + \beta_i V_i - \mathcal{L}_i V_i - f_i,\ V_i - \max_{j\ne i}(V_j - C_{ij})\} = 0$$

⇒ (two formulas)

③ $\underset{\downarrow}{\min}\left\{ \underline{-\partial_t V_i + \beta_i V_i - \mathcal{L}_i V_i - f_i},\ \underline{V_i - \max_{j\neq i}(V_j - C_{ij})} \right\} = 0$

∵ if it's "max"

by the definition of V_i,

$V_i \geq \max_{j\neq i}(V_j - C_{ij})$

if $V_i > \max_{j\neq i}(V_j - C_{ij})$, no switch, $-\partial_t V_i + \beta_i V_i - \mathcal{L}_i V_i - f_i = 0$

⇒ $\max\{ \cdots \} = V_i - \max_{j\neq i}(V_j - C_{ij}) > 0$, contradiction.

这启示我们 might think $V_i(t,x)$ 是某个 PDE 系统的解，∴

3.1 PDE system

$$\text{(3.1)}\begin{cases} \min\{\underbrace{-\partial_t U_i + \beta_i U_i - \mathcal{L}_i U_i - f_i}_{F_i(t,x,U_i,DU_i,D^2U_i)},\ U_i - \max_{j\neq i}(U_j - C_{ij})\} = 0, & (t,x)\in(0,T)\times O \\ \underline{U_i(T,x) = \psi_i(x)}, & x\in\bar{O} \end{cases}$$

↓ terminal condition

这里的 O 是一个有界区域，自然地，我们需要再加上边界条件的。

$\partial O := \bar{O}\setminus O$.

有两种边界条件：

Neumann boundary : $x\in\partial O$ $\frac{\partial U_i}{\partial \vec{n}} = \phi_i(t,x)$, $t\in[0,T)\times\partial O$ (3.2)

normal derivative; $\vec{n}$: the unit outward normarl vector of $\partial\Omega$ at x

Dirichlet ··· : $x\in\partial O$, $U_i(t,x) = d_i(t,x)$, $t\in[0,T)\times\partial O$ (3.3)

or Mixed boundary : $\begin{cases} \frac{\partial U_i}{\partial \vec{n}} = \phi_i(t,x), & t\in[0,T)\times\Gamma_1 \\ U_i(t,x) = d_i(t,x), & t\in[0,T)\times\Gamma_2 \end{cases}$, $\partial O = \underset{\text{open}}{\Gamma_1}\cup\underset{\text{closed}}{\Gamma_2}$ (3.4)

得到 (3.1) with (3.2) or with (3.3) 的解是光滑的，我们考虑更弱意义下的解. i.e.

Viscosity solution

Given a locally bounded function $u(t,x) : [0,T]\times\overline{\mathcal{O}} \to \mathbb{R}$ (i.e., for all $(t,x)\in[0,T]\times\overline{\mathcal{O}}$, there exists a compact neighborhood $V_{(t,x)}$ of (t,x) such that u is bounded on $V_{(t,x)}$), recall that the function u is lower (resp. upper)-semicontinuous at $(t,x)\in[0,T]\times\overline{\mathcal{O}}$ if for any $(s,y)\in[0,T]\times\overline{\mathcal{O}}$

$$\liminf_{(s,y)\to(t,x)} u(s,y) \geq u(t,x) \quad (\text{resp. } \limsup_{(s,y)\to(t,x)} u(s,y) \leq u(t,x)).$$

…, we denote it by $u\in LSC([0,T]\times\overline{\mathcal{O}})$ (resp. $u\in USC([0,T]\times\overline{\mathcal{O}})$). And we can define its lower-semicontinuous envelope u_* and upper-semicontinuous envelope u^* on $[0,T]\times\overline{\mathcal{O}}$ (see [3, pp.267, Definition 4.1]) by

→ Fleming-2006-book

$$u_*(t,x) := \liminf_{(s,y)\to(t,x)} u(s,y) := \sup_{r>0}\inf\{u(s,y)\in[0,T]\times\overline{\mathcal{O}}\cap B_r(t,x)\},$$

$$u^*(t,x) := \limsup_{(s,y)\to(t,x)} u(s,y) := \inf_{r>0}\sup\{u(s,y)\in[0,T]\times\overline{\mathcal{O}}\cap B_r(t,x)\},$$

which means that u_* (resp. u^*) is the largest (resp. smallest) lower-semicontinuous function (l.s.c.) ~~above~~ below (resp. upper-semicontinuous function (u.s.c.) ~~below~~ above) u on $[0,T]\times\mathcal{O}$. Then $u(t,x)$ is continuous at $(t,x)\in[0,T]\times\mathcal{O}$ iff $u(t,x) = u_*(t,x) = u^*(t,x)$

Next, we show the definition of viscosity solution as in [3, Remark 4.2 in pp.267] [8, Definition 4.2.1], which is equivalent to [5, Definition 2.3] [2, Definition 3.3] by applying Lemma 4.1 in [3, pp.211].

→ Hamadene-2023-JMAA

Definition 3.1 (Viscosity solution). *Let $u_i(t,x)$ $(i\in\Pi = \{1,2,3,\cdots,m\})$ be locally bounded. Then function $u(t,x) = (u_1,\cdots,u_m)(t,x)$ is a viscosity subsolution of* (3.1)-(3.2)*, if for all $i\in\Pi$ and any $W\in C^{1,2}([0,T]\times\overline{\mathcal{O}})$,*

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Olofsson-2022-AMO

$u^*(i) - W$ has a maximum, at $(\bar{t},\bar{x}) \in [0,T)\times\mathcal{O}$,

for Neumann boundary condition

$$\min[-\partial_t W + F_i(t,x,u_i^*,DW,D^2W), u_i^* - \max_{i\neq j}(u_j^* - c_{ij})] \leq 0$$

→ interior.

holds for all $(t,x) = (\bar{t},\bar{x}) \in [0,T)\times \mathcal{O}$ which is a maximum point of $u_i^*(t,x) - W(t,x)$;

(ii)

$$\min[-\partial_t W + F_i(t,x,u_i^*,DW,D^2W), u_i^* - \max_{i\neq j}(u_j^* - c_{ij})] \wedge [\partial_\nu u_i^* - \phi_i(t,x))] \leq 0$$

holds for all $(t,x) = (\bar{t},\bar{x}) \in [0,T)\times \partial\mathcal{O}$ which is a maximum point of $u_i^*(t,x) - W(t,x)$;

→ boundary.

(iii)

$$u_i^*(T,x) \leq \psi_i(x)$$

→ terminal condition

holds for all $x \in \overline{\mathcal{O}}$;

A viscosity supersolutions are defined analogously by replacing u_k^*, $\leq$, $\wedge$ as $(u_k)_*$, $\geq$, $\vee$, respectively, where $a\wedge b = min\{a,b\}$ and $a \vee b = max\{a,b\}$. A function is a viscosity solution of (3.1) - (3.2), *if it is both a viscosity subsolution and viscosity supersolution* (3.1) - (3.2).

并且在这个定义下，粘性解不一定要连续.

Def. 3.1' (Viscosity solution for Dirichlet boundary condition)

(ii) → $u_i^*(t,x) - d_i(t,x) \leq 0$ for all $(t,x) \in [0,T)\times\partial\mathcal{O}$.

Def. (Viscosity solution for mixed boundary condition)

(ii) →

(ii-1)

$$\min[-\partial_t W + F_i(t,x,u_i^*,DW,D^2W), u_i^* - \max_{i\neq j}(u_j^* - c_{ij})] \wedge [\partial_\nu u_i^* - \phi_i)] \leq 0$$

holds for all $(t,x) = (\bar{t},\bar{x}) \in [0,T)\times \Gamma_1$ which is a maximum point of $u_i^*(t,x) - W(t,x)$;

(ii-2)

$$u_i^*(t,x) - d_i(t,x) \leq 0$$

holds for all $(t,x) \in [0,T)\times \Gamma_2$;

lemma 3.3.

The value function $V_i(t,x)$ $(i \in \mathbb{I}_m)$ is the viscosity solution to (3.1) under definition 3.1 (i)(iii).

Proof. refer to Thm 5.2 in Kharroubi-2016-SIAM

or

refer to Pham-2008-book-P111

☆ If $V_i(t,x)$ satisfies the boundary condition (3.2) Neumann ... or (3.3) Dirichlet ......

in the sense of Def. 3.1 (ii)

\ (3.3) Dirichlet

in the sense of Def. 3.1 (ii) or, Def. 3.1' (ii)

Then by comparison principle $\Rightarrow$
$(V_i)_{i \in \mathbb{I}}$ is the unique viscosity solution of (3.1) with (3.2) or (3.1) with (3.3)

under the definition 3.1' or 3.1

References Thm 2.4 Thm 3.1

[Olofsson-2022-AMO or Hamadene-2023-JMAA (for Neumann boundary condition)
Pham-2008-book-pp.75 (for Dirichlet ⋯)]

Lemma 3.2 (Comparison principle)

Assumptions: ① $t \in [0,T]$,

$\mathcal{O} \subset \mathbb{R}^d$ $(d \geq 1)$ bdd open connected set.

smooth boundary $\partial\mathcal{O} = \bar{\mathcal{O}} \setminus \mathcal{O}$ satisfying some condition

($\bar{\mathcal{O}}$: closure of $\mathcal{O}$)

② $u = (u_i)_{1\le i\le m}$ — viscosity subsolution of (3.1)-(3.2) or (3.1)-(3.3)

$U = (U_i)_{1\le i \le m}$ — viscosity supersolution

(3.2): Neumann; (3.3): Dirichlet

Conclusion: for each $i \in \mathbb{I}_m$,

$u_i \leq U_i$, $\forall (t,x) \in [0,T) \times \bar{\mathcal{O}}$.

✓ for mixed boundary condition (3.4).

3.3 Optimal Strategy (Next time)

☆ (t,x,i) — initial condition (i: initial regime state)

☆ Recasting the optimal switching as iterative optimal stopping. > key idea

then using the Snell envelope to show that:

$(t,x) \in \{ \cdot\cdot V_i > \max \}$

The the sequence defined by

$$\tau_1^* := \inf\{ s \geq t : V_i = \max_{j \neq i} (V_j - C_{ij}) \} \wedge T$$

$$I_{\tau_1^*} = \sum_{j \in \mathbb{I}} k \cdot \chi_{\{\max_{j\neq i}(V_j - C_{ij}) = V_k(\tau_1^*, X_{\tau_1^*}^{t,x,i}) - C_{ik}(\tau_1^*, X_{\tau_1^*}^{t,x,i})\}}$$

$$=: \iota_1^*$$

$$\tau_2^* = \inf\{ s \geq \tau_1^* : V_{I_{\tau_1^*}} = \max_{\substack{j \neq i \\ \neq I_{\tau_1^*}}} (V_j - C_{I_{\tau_1^*} j}) \} \wedge T$$

⋯

$$\qquad\qquad j \neq i, \ \neq I_{\tau_1^*}$$

$I_{\tau_2^*} = \cdots$

$\cdots$

The $\alpha^* = (\tau_n^*, \iota_n^*)_{n\geq 1}$ is optimal stratedy, i.e. $J_i(t,x,\alpha^*) \geq J_i(t,x,\alpha)$ for all $\alpha \in \mathcal{A}$.

Hamadene - 2009 - SIAM, thm 1.
Ludkovski - 2005 - Phd thesis, pp. 17-18.

Some remarks on the definition of viscosity solution:

e.g., $\mathbb{I} = \{1,2\}$. h_{12}, h_{21} - switching cost

Define

complement set

$$C_{12} := \{(t,x) \in [0,T] \times [0,L] : u_1 - (u_2 - h_{12}) > 0\}$$
$$C_{21} := \{ \qquad : u_2 - (u_1 - h_{21}) > 0\}$$
$$S_{12} := \{(t,x) \in [0,T] \times [0,L] : u_1 = u_2 - h_{12}\}$$
$$S_{21} := \{ \qquad : u_2 = u_1 - h_{21}\}$$

$\therefore$ If (u_1, u_2) is viscocisty solution, by Definition $\Rightarrow$

① In C_{12}, $-\partial_t u_1 + \beta_1 u_1 - \mathcal{L}_1 u_1 - f_1 \leq 0$ (subsolution)

② everywhere, $-\partial_t u_1 + \beta_1 u_1 - \mathcal{L}_1 u_1 - f_1 \geq 0$ (supersolution)

①+② $\Rightarrow$ In C_{12}, viscocity solution u_1 is the classical solution of the following PDE system:

$$\begin{cases} -\partial_t u_1 + \beta_1 u_1 - \mathcal{L}_1 u_1 - f_1 = 0 \\ u_1 = u_2 - h_{12} \end{cases}$$

③ In S_{12}, $u_1 = u_2 - h_{12}$

$\Rightarrow$ everywhere, $\min\{-\partial_t u_1 + \beta_1 u_1 - \mathcal{L}_1 u_1 - f_1, \ u_1 - (u_2 - h_{12})\} \geq 0$ (u_1 super solution)

④ everywhere, $u_1 \geq u_2 - h_{12}$

③+④ $\Rightarrow$ viscosity supersolution also is the supersolution in classical sense ?