Optimal Strategy

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Recall

- An oil company operates accross three different production lines

$$\begin{cases} \text{crude oil} \longleftrightarrow \text{regime state } i=1, \\ \text{natural gas} \longleftrightarrow \quad i=2 \\ \text{refined oil} \longleftrightarrow \quad i=3 \end{cases}$$

X : price $\in [0, P]$; when $x = P$, stop producting.

- different production lines, X has different diffusion processes :
- control process : for any initial condition (t, x, i)

$$I_s^i := \sum_{n\geq 0} l_n 1_{[\tau_n, \tau_{n+1})}(s), \quad s \geq t, \quad \text{valued in } \{1, 2, 3\}.$$

with $\tau_0 = t$, $l_0 = i$, $t \in [0, T)$.

∴ control sequence $\alpha := (\tau_n, l_n)_{n\geq 1} \in A$

- diffusion process :

∴ whole diffusion process :

finite horizon

$$d X_s^{t,x,i} = \sum_{j=1}^{3} b_j \cdot 1_{\{j = I_s^i\}} \cdot ds + \sum_{j=1}^{3} \sigma_j \cdot 1_{\{j = I_s^i\}} \cdot dB_s, \quad t \leq s \leq T,$$

$0 \leq x \leq L$

$\exists!$ strong solution.

① $\begin{cases} b_1, \sigma_1 \rightarrow \text{drift / diffusion rate of } i=1 \text{ (crude oil)} \\ b_2, \sigma_2 \rightarrow \\ b_3, \sigma_3 \rightarrow \end{cases}$

② $X_s^{t,x,i}$: the diffusion process of X, from price x, time t and regime state i.

∴ subsystem

$$d X_s^{t,x,i} = b_{I_s^i} \, ds + \sigma_{I_s^i} \, dB_s, \quad \tau_n \leq s < \tau_{n+1}$$

- Objective function

$\tau := \inf\{ s \geq t : X_s^{t,x,i} = P \}$, → time stop producting.

$$\tau := \inf\{ s \geq t : X_s^{t,x,i} = P \}, \to$$ time stop producting.

and get terminal payoff $\psi_i(t, P)$.

$$J(t,x,i,\alpha) = E\Big[\int_t^{\tau\wedge T} e^{-\beta_{I_s^i}(s, X_s^{t,x,i})(s-t)} f_{I_s^i}(s, X_s^{t,x,i})\, ds$$ → running

expected payoff

$\alpha = (\tau_n, \iota_n)_{n\geq 1}$

$$- \sum_{t\leq \tau_n} e^{-\beta_{I_s^i}(s, X_s^{t,x,i})(\tau_n - t)} h_{\iota_{n-1}, \iota_n}(s, X_s^{t,x,i}) \cdot 1_{\{\tau_n < \tau\wedge T\}}$$ → switching cost

$$+ e^{-\beta_{I_\tau^i}(\tau, X_\tau^{t,x,i})(\tau - t)} \cdot \psi_{I_\tau^i}(\tau, P) \cdot 1_{\{\tau \leq T\}}\Big]$$ → terminal payoff

$\boxed{\text{aim}}$ Find the optimal strategy α^* s.t.

$$J(t,x,i,\alpha^*) \geq J(t,x,i,\alpha) \text{ for all } \alpha \in A,$$

i.e., optimal

$$(1) \qquad J(t,x,i,\alpha^*) = \sup_{\alpha\in A} J(t,x,i,\alpha) =: \underline{V_i(t,x)}.$$ → value function

By the def. of $V_i(t,x)$, $\Rightarrow$

$$\begin{cases} V_i(T,x) = 0, & x \in [0,P) \to \text{terminal} \\ V_i(t,P) = \psi_i(t,P), & t \in [0,T],\ x = P \to \text{boundary} \end{cases}$$

二. • Hamilton-Jacobi-Bellman (HJB in short) equation

上次，已介绍:

$V_i(t,x) \in C([0,T)\times[0,L))$ is the unique viscosity solution of HJB variational inequality system:

$$\begin{cases} \min\{ -\partial_t V_i + \beta_i V_i - \mathcal{L}_i V_i - f_i,\ V_i - \max_{j\neq i}(V_j - h_{ij}) \} = 0, & (t,x) \in (0,T)\times(0,P) \\ V_i(t,P) = \psi_i(t,P), & t\in[0,T],\ x = P \\ V_i(T,x) = 0 & x \in [0,P) \\ \partial_x V_i(t,x) = 0 & t \in [0,T],\ x = 0 \end{cases}$$

Aim: find the optimal strategy.

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因为 V_i 是上述 PDE 系统的 黏性解,

∴ 我们

三. Optimal Stratedy.

1. Define

① $$S_{ij} = \{ (t,x) \in [0,T] \times [0,P] : V_i = V_j - h_{ij} \} \quad (3.1)$$

$$S_i = \{ \quad : V_i = \max_{j \neq i} (V_j - h_{ij}) \} \quad (3.2)$$

⇒

$S_i = \bigcup_{j \neq i} S_{ij}$, $i = \{1,2,3\}$ — switching region, where change the regime i.

②

The complement set C_i of S_i in $[0,T] \times [0,P]$ is

$$C_i = \{ \quad : V_i > \max_{j \neq i} (V_j - h_{ij}) \}, \quad (3.3)$$

where stay in regime i.

Key goal: the optimal strategy can be described by (3.1) - (3.3).

2. 首先,形式讨论一个简单的 case:

• Formal discussion $\quad \tau := \inf \{ s \geq t : X_s^{t,x,i} = P \}$

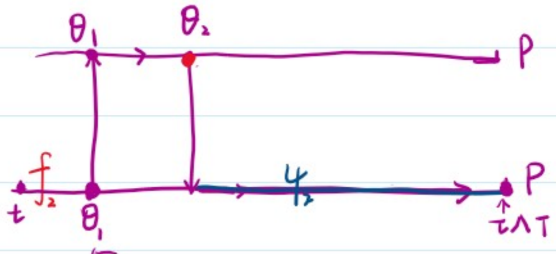

Figure 1

$t \leq \theta_i \leq \tau \wedge T$

stopping time for regime state 2's diffusion

∴ optimal switching problem → a recursive sequence of stopping problem

⇒ ↗ switch 2 times

$\Rightarrow$ switch 2 times

$$J^2(t,x,2) = E\Big[\int_t^{\theta_1} f_2(s, X_s^{t,x,2})ds$$
$$+ \left(J^1(\theta_1, X_{\theta_1}^{t,x,1}, 1) - h_{21}(\theta_1)\right)\cdot 1\{\theta_1 < \tau\wedge T\}$$
$$+ \psi_1(\tau, P)\cdot 1\{\theta_1 = \tau \le T\}$$

where

$$J^1(\theta_1, X_{\theta_1}^{t,x,2}, 1) = E\Big[\int_{\theta_1}^{\theta_2} f_1(s, X_s^{t,x,1})$$
$$+\left(\int_{\theta_2}^{\tau\wedge T} f_2(s, X_s^{t,x,2}) - h_{12}(\theta_2)\right)\cdot 1\{\theta_2 < \tau\wedge T\}$$
$$+ \psi_2(\tau, P)\cdot 1\{\theta_2 = \tau \le T\}$$

这鼓励考虑更一般的 regime state & switching times.

$\Rightarrow$ 引入以下迭代序列：

For any $i \in \{1,2,3\}$, $0 \le t \le T$, define

$$Y^0(t,x,i)$$
$$:= E\left[\int_t^{\tau\wedge T} e^{-\beta_i} f_i(s, X_s^{t,x,i})ds + e^{-\beta_i(\tau, X_\tau^{t,x,i})}\psi_i(\tau,P)\cdot 1_{\{\tau\le T\}} \,\Big|\, \mathcal{F}_t\right]$$

for $n \ge 1$,

$$Y^n(t,x,i)$$
$$:= \operatorname*{ess\,sup}_{\theta\in\mathcal{T}_{t,\tau\wedge T}} E\Big[\int_t^{\theta} e^{-\beta_i} f_i(s, X_s^{t,x,i})ds$$
$$+ \max_{k\ne i}\left(Y^{n-1}(\theta, X_\theta^{t,x,i}, k) - e^{-\beta_i(\theta, X_\theta^{t,x,i})}\cdot h_{ik}(\theta)\right)\cdot 1_{\{\theta<\tau\wedge T\}}\Big]$$

$t \le \theta \le \tau\wedge T$, stopping time

$$+ \psi_{I_\tau^i}(\tau, P)\cdot 1_{\{\theta=\tau\le T\}} \,\Big|\, \mathcal{F}_t\Big]$$

3. 那么对于这样定义的迭代序列，有很好的性质：

Prop. 3 (See Hamadene - 2009 - SIAM J.C.O)

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① The limit process

$$Y(t,x,i) := \lim_{n\to\infty} Y^n(t,x,i) \text{ exists},$$

its right continuous and $\exists$ left limit about t (i.e., càdlàg process)

② And these limit processes

$$Y^i := \left(Y(t,x,i) \right)_{0\le t\le T}$$

fulfill

→ L^q- integrable.

(a) $E\left[\sup_{0\le t\le T} |Y(t,x,i)|^q \right] < \infty$, $i=1,2,3$, $q \ge 1$,

(b) $\forall\ 0\le t\le T$,

$$\begin{aligned} Y(t,x,i) &= \operatorname*{ess\,sup}_{\theta\in \mathcal{T}_{t,\tau\wedge T}} E\Big[\int_t^\theta f_i(s, X_s^{t,x,i})ds \\ &\quad + \max_{k\ne i} \left(Y(\theta, X_\theta^{t,x,i}, k) - h_{ik}(\theta) \right)\cdot 1_{\{\theta<\tau\wedge T\}} \\ &\quad + \Psi_{I_\tau^i}(\tau, p)\cdot 1_{\{\theta=\tau\le T\}} \Big| \mathcal{F}_t \Big\} \qquad (2) \end{aligned}$$

for convenience, omit the exponential term.

Moreover, $Y^i := (Y(t,x,i))_{0\le t\le T}$ are continuous.

(see thm 2. in Hamadene - 2009 - SIAM J. C. O.)

4. 那么有了表达式 (2)，我们可以利用 Snell envelope 的性质证明

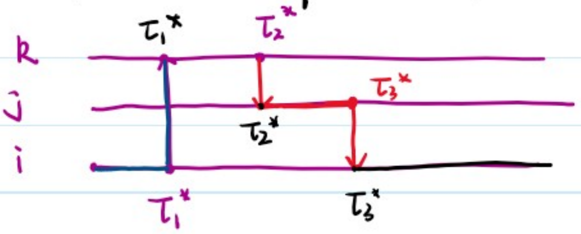
Thm. ☆

① $\underset{(2)}{Y(t,x,i)} = \sup_{\alpha\in A} J(t,x,i) = \underset{(1)}{V(t,x,i)}$

$\Rightarrow$

$\Rightarrow$

② Define the sequence of $\mathbb{F}$-stopping times $(\tau_n^*)_{n\ge1}$ by



$$\tau_1^* := \inf\{ s \ge t : V_i = \max_{j\ne i} (V_j - h_{ij})(s, X_s^{t,x,i})\} \wedge \tau \wedge T$$

$$\tau_n^* := \inf\{ s \ge \tau_{n-1}^* : V_{I^i_{\tau^*_{n-1}}} = \max (V_k - h_{I^i_{\tau^*_{n-1}} k})(s, X_s^{t,x,I^i_{\tau^*_{n-1}}})\} \wedge \tau \wedge T$$

$$\underset{\iota_1}{\overset{!!}{I^i_{\tau_1^*}}} = \sum_{k\in\{1,2,3\}} k \cdot \mathbb{1}\{\max_{j\ne i}(V_j - h_{ij})(\tau_1^*, X^{t,x,i}_{\tau_1^*}) = (V_k - h_{I^i_{\tau_1^*}k})(\tau_1^*, X^{t,x,i}_{\tau_1^*})\}$$

$n \ge 2$,

$I^i_{\tau_n^*} = \iota_n$ can be decided by the set

$$\left\{ \max_{j \ne I^i_{\tau^*_{n-1}}} (V_j - h_{I^i_{\tau^*_{n-1}} j})(\tau_n^*, X^{t,x,i}_{\tau_n^*}) = (V_l - h_{I^i_{\tau^*_{n-1}} l})(\tau_n^*, X^{t,x,i}_{\tau_n^*}) \right\}$$

Then (τ_n^*, ι_n^*) is the optimal strategy.

Remark: $(\tau_n^*)_{n\ge1} \Leftrightarrow$ switching region

Proof.

Key idea: by Snell envelope $\Rightarrow$ optimal stopping time.

首先，给出 Snell envelope 的定义和重要性质.

We introduce and summarize some important results on Snell envelopes, which play a key role in optimal stopping problems. For $t \in [0, T]$, $T < \infty$, we denote by $\mathcal{T}_{t,T}$ the set of stopping times valued in $[t, T]$.

Pham-book-pp.8 or Hamadene-2009-SIAM. prop.2

set of stopping time 0->T

Proposition 1.1.8 *(Snell envelope)* *Let $H = (H_t)_{0\le t\le T}$ be a real-valued $\mathbb{F}$-adapted càd-làg process in the class (DL). The*

Proposition 1.1.8 *(Snell envelope)* Pham-book-pp.8 or Hamadene-2009-SIAM. prop.2

set of stopping time 0->T

Let $H = (H_t)_{0 \le t \le T}$ be a real-valued $\mathbb{F}$-adapted càd-làg process, in the class (DL). The Snell envelope V of H is defined by

右连续并左极限存在

i.e., $\{H_\tau, \tau \in \mathcal{T}_0\}$ is uniformly integrable

$$V_t = \operatorname*{ess\ sup}_{\tau \in \mathcal{T}_{t,T}} E[H_\tau | \mathcal{F}_t], \quad 0 \le t \le T,$$

and it is the smallest supermartingale of class (DL), which dominates H: $V_t \ge H_t$, $0 \le t \le T$. Furthermore, if H has only positive jumps, i.e. $H_t - H_{t^-} \ge 0$, $0 \le t \le T$, then V is continuous, and for all $t \in [0, T]$, the stopping time

$\Uparrow$

$E[\sup_{0 \le s \le T} |H_s|] < \infty, \forall \tau \in \mathcal{T}_0$

$V = (V_t)_{0 \le t \le T}$

$$\tau_t = \inf\{s \ge t : V_s = H_s\} \wedge T$$

is optimal after t, i.e.

$$V_t = E[V_{\tau_t} | \mathcal{F}_t] = E[H_{\tau_t} | \mathcal{F}_t].$$

$$= \operatorname*{ess\ sup}_{\tau \in \mathcal{T}_{t,T}} E[H_\tau | \mathcal{F}_t], \quad 0 \le t \le T.$$

为什么这样定义的 τ_t 是 optimal stopping time？形式上可以这样理解：

H_s：即时收益

V_s：表示从 s 时刻开始，[redacted]，随机过程能达到的最大收益期望

∴ $H_s = V_s$：表示立即停止的收益 = 未来期望的最优收益，没必要继续等待.

∴ 我们的问题就找到合适的 process H 以及 Snell envelop V，就可以解决了.

Step1. We already had the existence of

$$Y(t,x,i) = \operatorname*{ess\ sup}_{\theta \in \mathcal{T}_{t,\tau \wedge T}} E\Big[\int_t^\theta f_i(s, X_s^{t,x,i}) ds + \max_{k \ne i} \big(Y(\theta, X_\theta^{t,x,i}, k) - h_{ik}(\theta) \big) \cdot 1_{\{\theta < \tau \wedge T\}}$$

$$+ \ H_i(T, D) \cdot 1_{\{\ldots\}} \ \ldots \Big] \quad (2)$$

$$k \neq i$$

$$+ \Psi_{I^i_\tau}(\tau, P) \cdot \mathbb{1}_{\{\theta = \tau \le T\}} \Big| \mathcal{F}_t \Big\} \qquad (2)$$

like the form of Snell envelop

$\Leftrightarrow$

$$Y(t,x,i) + \int_0^t f_i(s, X_s^{t,x,i})\,ds \qquad, 0 \le t \le T$$

$$= \operatorname*{ess\,sup}_{\theta \in \mathcal{I}_{t,\tau\wedge T}} E\Big[\int_0^\theta f_i(s, X_s^{t,x,i})\,ds$$

$$+ \max_{k \neq i} \big(Y(\theta, X_\theta^{t,x,i}, k) - h_{ik}(\theta) \big) \cdot \mathbb{1}_{\{\theta < \tau\wedge T\}}$$

$$+ \Psi_{I^i_\tau}(\tau, P) \cdot \mathbb{1}_{\{\theta = \tau \le T\}} \Big| \mathcal{F}_t \Big\} \qquad (2)$$

$$\Leftrightarrow V_t = \operatorname*{ess\,sup}_{\theta \in \mathcal{I}_{t,\tau\wedge T}} E[H_\theta | \mathcal{F}_t]$$

Satisfy the conditions in Prop. 1.18

$\Rightarrow V = (V_t)_{0 \le t \le T}$ is the Snell envelope of $H = (H_t)_{0 \le t \le T}$.

$\Rightarrow$ By prop. 1.1.8,

$\therefore$ $\tau_1^* := \inf\{ s \ge t : V_s = H_s \}$ is the optimal stopping time,

where

$$H_s = \cancel{\int_0^s f_i\,dz}$$

$$+ \max_{k \neq i} \big(Y(s, X_s^{t,x,i}, k) - h_{ik}(s, X_s^{t,x,i}) \big) \cdot \mathbb{1}_{\{s < \tau\wedge T\}}$$

$$+ \Psi_{I^i_\tau}(\tau, P) \cdot \mathbb{1}_{\{\ldots\}} \quad \ldots \le T$$

$$+ \underbrace{\psi_{I^i_\tau}(\tau, p) \cdot \mathbb{1}_{\{s = \tau \le T\}} \}}_{\text{no switch}}, \quad \forall \quad t \le s \le T$$

$$V_s = Y(s, x, i) + \cancel{\int_0^s f_i \, dz}$$

$$\Rightarrow \tau_1^* := \inf\{ s \ge t : \; Y(s, X_s^{t,x,i}, i) = \max_{k \ne i} \left(Y(s, X_s^{t,x,i}, k) - h_{ik}(s, X_s^{t,x,i}) \right) \} \wedge \tau \wedge T$$

is the optimal stopping time.

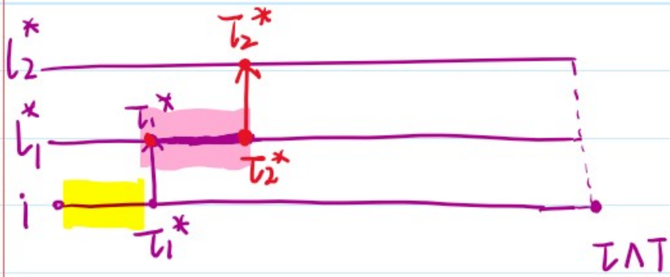
$$I^i_{\tau_1^*} := \sum_{k \in \{1,2,3\}} k \cdot \mathbb{1}\left\{ \max_{j \ne i} (V_j - h_{ij})(\tau_1^*, X^{t,x,i}_{\tau_1^*}) = (V_k - h_{I^i_{\tau_1^*} k})(\tau_1^*, X^{t,x,i}_{\tau_1^*}) \right\}$$

$$\text{!!} \quad l_1^* \in \{1,2,3\}$$

$$\Rightarrow Y(s, x, i)$$

$$\overset{(2)}{=} \operatorname*{ess\,sup}_{\theta \in \mathcal{T}_{t, \tau \wedge T}} E\Big[\int_t^\theta f_i(s, X_s^{t,x,i}) ds + \max_{k \ne i} \left(Y(\theta, X_\theta^{t,x,i}, k) - h_{ik}(\theta) \right) \cdot \mathbb{1}_{\{\theta < \tau \wedge T\}} + \psi_{I^i_\theta}(\theta, p) \cdot \mathbb{1}_{\{\theta = \tau \le T\}} \Big| \mathcal{F}_t \Big\} \qquad (2)$$

$$= E\Big[\int_t^{\tau_1^*} f_i(s, X_s^{t,x,i}) ds + \left(Y(\tau_1^*, X^{t,x,i}_{\tau_1^*}, I^i_{\tau_1^*}) - h_{i I^i_{\tau_1^*}}(\tau_1^*) \right) \cdot \mathbb{1}_{\{\tau_1^* < \tau \wedge T\}} + \psi_{I^i_\tau}(\tau, p) \cdot \mathbb{1}_{\{\tau_1^* = \tau \le T\}} \Big| \mathcal{F}_t \Big] \qquad (\text{☆}1)$$



step 2 $\forall \; \tau_1^* \le t \le T$ we can show

τ_1^* $\quad\quad$ $\tau\wedge T$

step 2. $\forall\ \tau_1^* \le t \le T$, we can show

$$Y(t, x, \overset{l_1^*}{\overset{\shortparallel}{I^i_{\tau_1^*}}}) + \int_{\tau_1^*}^{t} f_{l_1^*}\, ds$$

$$= \operatorname*{ess\,sup}_{\theta\in\mathcal{T}_{t,\tau\wedge T}} E\Big[\int_t^\theta f_{l_1^*}(s, X_s^{t,x,i})\,ds + \int_{\tau_1^*}^{t} f_{l_1^*}\cdot ds$$

$$+ \max_{k\neq l_1^*}\big(Y(\theta, X_\theta^{t,x,i}, k) - h_{l_1^*k}(\theta)\big)\cdot \mathbb{1}_{\{\theta<\tau\wedge T\}}$$

$$+ \Psi_{I^i_\tau}(\tau, P)\cdot \mathbb{1}_{\{\theta=\tau\le T\}}\ \Big|\ \mathcal{F}_t\Big]$$

(see proof of thm 1 in Hamadene - 2009 - SIAM J.C.O.)

$$\Longleftrightarrow \underbrace{Y(s, x, \overset{l_1^*}{\overset{\shortparallel}{I^i_{\tau_1^*}}}) + \int_{\tau_1^*}^{s} f_{l_1^*}\, ds}_{U_s},\ t\ge\tau_1^*,\ \text{Snell envelope of}$$

$$\underbrace{\boxed{\int_{\tau_1^*}^{s} f_{l_1^*}\,dz + \max_{k\neq l_1^*}\big(Y(s, X_s^{t,x,i}, k) - h_{l_1^*k}(s)\big)\cdot\mathbb{1}_{\{s<\tau\wedge T\}} + \Psi_{I^i_\tau}(\tau, P)\cdot \mathbb{1}_{\{s=\tau\le T\}}}}_{G_s}$$

$\therefore$

$$\tau_2^* := \inf\{s\ge\tau_1^*: U_s = G_s\}\wedge\tau\wedge T$$

$$= \inf\{s\ge\tau_1^*: Y(s, X_s^{t,x,i}, \overset{l_1^*}{\overset{\shortparallel}{I^i_{\tau_1^*}}}) = \max_{k\neq l_1^*}\big(Y(s, X_s^{t,x,i}, k) - h_{l_1^*k}(s)\big)\}\wedge\tau\wedge T$$

By prop.1-8, $\Rightarrow \tau_2^*$ is the optimal sto

$\Rightarrow$ $\quad\quad$ $\overset{l_1^*}{\shortparallel}$

$\Rightarrow$

$$Y(\tau_1^*, x, \overset{l_1}{\overset{\shortparallel}{I^i_{\tau_1^*}}})$$

$$= E\Big[\int_{\tau_1^*}^{\tau_2^*} f_{l_1^*}(s, X_s^{t,x,l_1^*})ds$$

$$+ \Big(Y(\tau_2^*, X_{\tau_2^*}^{t,x,l_1}, \overset{l_2^*}{\overset{\shortparallel}{I^i_{\tau_2^*}}}) - h_{l_1^* l_2^*}(\tau_2^*)\Big) \cdot 1_{\{\tau_2^* < \tau \wedge T\}}$$

$$+ \psi_{I^i_\tau}(\tau, p) \cdot 1_{\{\tau_2^* = \tau \le T\}} \,\Big|\, \mathcal{F}_t \Big],$$

substituted into (☆1) $\Rightarrow$

$$Y(t, x, i)$$

$$= E\Big[\int_t^{\tau_1^*} f_i(s, X_s^{t,x,i})ds$$

$$+ \Big(Y(\tau_1^*, X_{\tau_1^*}^{t,x,i}, I^i_{\tau_1^*}) - h_{i I^i_{\tau_1^*}}(\tau_1^*)\Big) \cdot 1_{\{\tau_1^* < \tau \wedge T\}}$$

$$+ \psi_{I^i_\tau}(\tau, p) \cdot 1_{\{\tau_1^* = \tau \le T\}} \,\Big|\, \mathcal{F}_t \Big] \quad (☆1)$$

$$= E\Big[\int_t^{\tau_1^*} f_i \, ds - h_{i l_1^*}(\tau_1^*) \cdot 1_{\{\tau_1^* < \tau \wedge T\}}$$

$$+ \int_{\tau_1^*}^{\tau_2^*} f_{l_1^*} \, ds \cdot 1_{\{\tau_1^* < \tau \wedge T\}}$$

$$- h_{l_1^* I^i_{\tau_2^*}}(\tau_2^*) \cdot 1_{\{\tau_2^* < \tau \wedge T\}} \cdot 1_{\{\tau_1^* < \tau \wedge T\}}$$

$$+ \psi_{I^i_\tau}(\tau, p) \cdot 1_{\{\tau_1^* = \tau \le T\}} \quad = 0 \;\because\; \tau_1^* < \tau_2^* = \tau \le T$$

$$+ \psi_{I^i_\tau}(\tau, p) \cdot 1_{\{\tau_2^* = \tau \le T\}} \cdot 1_{\{\tau_1^* < \tau \wedge T\}}$$

$$+ Y(\tau_2^*, X_{\tau_2^*}^{t,x,i}, l_2^*) \cdot 1_{\{\ldots\}} \cdot 1 \ldots$$

$$+ Y(\tau_2^*, X_{\tau_2^*}^{t,x,i}, l_2^*) \cdot 1_{\{\tau_2^* < \tau \wedge T\}} \cdot 1_{\{\tau_1^* < \tau \wedge T\}}$$

$\{\tau_2^* < \tau \wedge T\} \subset \{\tau_1^* < \tau \wedge T\}$

$$= E[\int_t^{\tau_2^*} f_{I_s^i}(s, X_s^{t,x,i}) ds$$

$$- h_{i l_1^*}(\tau_1^*) \cdot 1_{\{\tau_1^* < \tau \wedge T\}} - h_{l_1^* l_2^*}(\tau_2^*) \cdot 1_{\{\tau_2^* < \tau \wedge T\}}$$

$$+ Y(\tau_2^*, X_{\tau_2^*}^{t,x,i}, l_2^*) \cdot 1_{\{\tau_2^* < \tau \wedge T\}}$$

$$+ \psi_{I_\tau^i}(\tau, p) \cdot 1_{\{\tau_2^* = \tau \leq T\}}$$

Step3. 类似于 step 2，将橙色项展开，repeating n times $\Rightarrow$

$$Y(t,x,i) = E[\int_t^{\tau_n^*} f_{I_s^i}(s, X_s^{t,x,i}) ds$$

$$- \sum_{t \leq \tau_j}^{n} h_{l_{j-1}^* l_j^*}(\tau_j^*) \cdot 1_{\{\tau_j^* < \tau \wedge T\}}$$

$$+ Y(\tau_n^*, X_{\tau_n^*}^{t,x,i}, l_n^*) \cdot 1_{\{\tau_n^* < \tau \wedge T\}}$$

$$+ \psi_{I_\tau^i}(\tau, p) \cdot 1_{\{\tau_n^* = \tau \leq T\}}]$$

let $n \to \infty$.

$$Y(t,x,i) = E[\int_t^{\tau \wedge T} f_{I_s^i} ds - \sum_{t \leq \tau_j} h_{l_{j-1}^* l_j^*}(\tau_j^*) \cdot 1_{\{\tau_j^* < \tau \wedge T\}}$$

$$+ \psi_{I^i}(\tau, p) \cdot 1 \ldots$$

$$+ \ \Psi_{I_\tau^i}(\tau, p) \cdot \mathbb{1}_{\underbrace{\{\tau \wedge T = \tau\}}_{\{\tau \leq T\}}}$$

$$= J(t, x, i, \alpha^*)$$

Step 4. $J(t,x,i,\alpha^*) \geq J(t,x,i,\alpha), \ \forall \alpha \in \mathcal{A}, \ \alpha = (\tau_n, l_n)_{n \geq 1}$

①

$$Y(t,x,i) \overset{\text{Snell envelope definition}}{\geq} E\Big[\int_t^{\tau_1} f_i \, ds + \max_{j \neq i}\big(Y(\tau_1, X_{\tau_1}^{t,x,i}, j) - h_{ij}(\tau_1)\big) \cdot \mathbb{1}_{\{\tau_1 < \tau \wedge T\}}$$

$$+ \Psi_{I_\tau^i}(\tau, p) \cdot \mathbb{1}_{\{\tau_1 = \tau \leq T\}}\Big]$$

$$\geq E\Big[\int_t^{\tau_1} f_i \, ds + \big(Y(\tau_1, X_{\tau_1}^{t,x,i}, l_1) - h_{il_1}(\tau_1)\big) \cdot \mathbb{1}_{\{\tau_1 < \tau \wedge T\}}$$

$$+ \Psi_{I_\tau^i}(\tau, p) \cdot \mathbb{1}_{\{\tau_1 = \tau \leq T\}}\Big]$$

② Similarly,

$$Y(\tau_1, X_{\tau_1}^{t,x,i}, l_1) \geq E\Big[\int_{\tau_1}^{\tau_2} f_{l_1} \cdot ds + \max_{j \neq l_1}\big(Y(\tau_2, X_{\tau_2}^{t,x,i}, j) - h_{l_1 j}(\tau_2)\big) \cdot \mathbb{1}_{\{\tau_2 < \tau \wedge T\}}$$

$$+ \Psi_{I_\tau^i}(\tau, p) \cdot \mathbb{1}_{\{\tau_2 = \tau \leq T\}},$$

$$\geq E\Big[\int_{\tau_1}^{\tau_2} f_{l_1} \cdot ds + \big(Y(\tau_2, X_{\tau_2}^{t,x,i}, l_2) - h_{l_1 l_2}(\tau_2)\big) \mathbb{1}_{\{\tau_2 < \tau \wedge T\}}$$

$$+ \Psi_{I_\tau^i}(\tau, p) \cdot \mathbb{1}_{\{\tau_2 = \tau \leq T\}},$$

代入上式

$\Rightarrow$

$$Y(t,x,i) \geq E\Big[\int^{\tau_2} f_i \, ds - h_{il_1}(\tau_1) \cdot \mathbb{1}_{\{\ldots\}} - h_{l_1 l_2}(\tau_2) \cdot \mathbb{1}_{\{\ldots\}}$$

$\Rightarrow$

$$Y(t,x,i) \geq E\Big[\int_t^{\tau_2} f_{I_s^i}\, ds - h_{i l_1}(\tau_1)\cdot 1_{\{\tau_1 < \tau\wedge T\}} - h_{l_1 l_2}(\tau_2)\cdot 1_{\{\tau_2 < \tau\wedge T\}}$$

$$+ \Psi_{I_\tau^i}(\tau, p)\cdot 1_{\{\tau_2 = \tau \leq T\}}\Big]$$

.... 重复n次并且 let $n \to \infty$

$$Y(t,x,i) \geq E\Big[\int_t^{\tau\wedge T} f_{I_s^i}(s, X_s^{t,x,i})\, ds - \sum_{t \leq \tau_j} h_{l_{j-1} l_j}(\tau_j)\cdot 1_{\{\tau_j < \tau\wedge T\}}$$

$$+ \Psi_{I_\tau^i}(\tau, p)\cdot 1_{\{\tau\wedge T = \tau\}}\Big]$$

$$\{\tau\wedge T = \tau\} = \{\tau \leq T\}$$

$\Rightarrow J(t,x,\alpha^*) \geq J(t,x,\alpha), \ \forall \alpha \in A.$

$\therefore$ α^* is the optimal strategy. and

$Y(t,x,i) = V_i(t,x)$

$\Rightarrow$ S_i, C_i can describe the optimal strategy. #