Devoir 5

Instructeur: King-Yeung Lam

le 14 fevrier, 2022

Consider the competition system

$$\begin{cases} u_t - d_1\Delta u = u(m(x) - u - v) & \text{in } \Omega \times (0,\infty), \\ v_t - d_2\Delta v = v(m(x) - u - v) & \text{for } \Omega \times (0,\infty), \\ \text{Neumann b.c.} & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0, \quad v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{0.1}$$

Suppose also that $\int_\Omega m(x) > 0$, so that (0.1) has at least three equilibria:

$$E_0 = (0,0), \quad E_1 = (\tilde{u}, 0), \quad E_2 = (0, \tilde{v}).$$

Linearizing at the equilibrium E_2, we obtain

$$\begin{cases} d_1\Delta\varphi + (m(x) - \tilde{v})\varphi + \lambda\varphi = 0 & \text{in } \Omega, \\ d_2\Delta\psi - \tilde{v}\varphi + (m(x) - 2\tilde{v})\psi + \lambda\psi = 0 & \text{in } \Omega, \\ \text{Neumann b.c.} & \text{on } \partial\Omega. \end{cases} \tag{0.2}$$

Definition 0.1. For $(d,h) \in (0,\infty) \times C(\overline{\Omega})$, define $\mu(d,h)$ to be the principal eigenvalue of

$$\begin{cases} d\Delta\phi + h\phi + \mu\phi = 0 & \text{in } \Omega, \\ \text{Neumann b.c.} & \text{on } \partial\Omega. \end{cases} \tag{0.3}$$

1. Suppose $m(x)$ is nonconstant, and $(\hat{u}(x), \hat{v}(x))$ is a positive equilibrium of (0.1). Show that $m - \hat{u} - \hat{v}$ is nonconstant.

2. Justify $\mu(d_2, m - \tilde{v}) = 0$.

3. Let $\tilde{\mu}$ be the principal eigenvalue of

$$\begin{cases} d_2\Delta\phi + (m(x) - 2\tilde{v})\phi + \mu\phi = 0 & \text{in } \Omega, \\ \text{Neumann b.c.} & \text{on } \partial\Omega, \end{cases} \tag{0.4}$$

   Show that $\tilde{\mu} > 0$.

4. Show that $\mathcal{L}_\mu = -d_2\Delta - (m - 2\tilde{v}) - \mu I$ is invertible for all $\mu \le 0$. [Hint: Consider the sign of the principal eigenvalue. It is the eigenvalue with the positive real part.]

5. Suppose the principal eigenvalue μ_{E_2} of

$$\begin{cases} d_1\Delta\phi + (m(x) - \tilde{v})\phi + \mu\phi = 0 & \text{in } \Omega, \\ \text{Neumann b.c.} & \text{on } \partial\Omega, \end{cases} \tag{0.5}$$

satisfies $\mu_{E_2} \le 0$, show that μ_{E_2} is also an eigenvalue of (0.2) by constructing the corresponding eigenfunction.

6. Consider the following system of ordinary differential equations.

$$x_1' = x_1(1 - x_1 - x_2), \quad x_2' = x_2(1 - \mu x_1 - x_2)^3. \tag{0.6}$$

Suppose $\mu > 1$. Show that the semiflow generated by (0.6) satisfies (H1')-(H4') and has no positive equilibrium; (iii) $E_2 = (0, 1)$ is locally asymptotically stable. Moreover, $E_1 = (1, 0)$ attracts some internal trajectories. [Hint: E_1 attracts all trajectories initiating in $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 1, 0 < x_2 < (x_1 - 1)^2\}$.]