Population structured by space & trait.

[Perthame-Souganidis (2016); Lam-Lou (2017)]

$u = u(x, z, t)$, $x \in D$, $z \in \mathbb{R}$, $t > 0$

$$\begin{cases} \partial_t u = \Theta(z)\Delta_x u + u\left(m(x) - \int_0^1 u(x,z,t)dz\right) + \varepsilon^2 \partial_{zz} u & x \in D,\ z \in \mathbb{R},\ t > 0. \\ n \cdot \nabla_x u = 0 & x \in \partial D,\ z \in \mathbb{R},\ t > 0 \\ u(x,z,t) = u(x, z+1, t) \end{cases}$$

We will analyze the equilibrium solution. $u = u_\varepsilon(x,z)$

$$(1)\begin{cases} \Theta(z)\Delta_x u_\varepsilon + u_\varepsilon(m(x) - \rho_\varepsilon(x)) + \varepsilon^2 \partial_{zz} u = 0 & (x,z) \in D \times \mathbb{R} \\ n \cdot \nabla_x u_\varepsilon = 0 & (x,z) \in \partial D \times \mathbb{R} \\ u_\varepsilon(x,z) = u_\varepsilon(x,z+1) & (x,z) \in D \times \mathbb{R} \end{cases}$$

where $\rho_\varepsilon(x) = \int_0^1 u_\varepsilon(x,z)dz$.

__Theorem__. Suppose ① $m(x)$ is positive, nonconst.

② $\Theta(z)$ is cont., positive and 1-periodic.

③ there exists a unique $\hat{z} \in [0,1]$ st.

$$\inf_{\mathbb{R}} \Theta = \Theta(\hat{z}).$$

Let u_ε be any positive sol. of (1), then as $\varepsilon \to 0$,

$$u_\varepsilon(x,z) \longrightarrow \delta_0(z-\hat{z})\,\hat{\theta}(x) \quad \text{in distribution sense in } D\times[0,1],$$

where $\hat{\theta}(x)$ is the unique positive sol. of

$$\Theta(\hat{z})\Delta\hat{\theta} + \hat{\theta}(m(x) - \hat{\theta}) = 0 \text{ in } D \quad n\cdot\nabla\hat{\theta} = 0 \text{ on } \partial D.$$

Lemma 1 For $\varepsilon > 0$ small,

$$\|\rho_\varepsilon\|_{W^{2,p}(D)} \leq C$$

and

$$0 < \frac{1}{2}\inf_D m \leq \rho_\varepsilon(x) \leq 2\sup_D m \quad \text{in } x \in D.$$

Pf First, we show $\|\rho_\varepsilon\|_{C(\bar{D})} \leq C$.

$$\Delta_x \rho_\varepsilon + \varepsilon^2 P_\varepsilon + Q_\varepsilon(m(x) - \rho_\varepsilon) = 0 \text{ in } D, \quad n\cdot\nabla\rho_\varepsilon \text{ in } \partial\Omega.$$

where

$$P_\varepsilon(x) = \int_0^1 \frac{1}{\Theta(z)}\partial_{zz}u_\varepsilon\,dz = \int_0^1 \left(\frac{1}{\Theta(z)}\right)_{zz} u_\varepsilon\,dz$$

$$Q_\varepsilon(x) = \int_0^1 \frac{1}{\Theta(z)} u_\varepsilon\,dz$$

Note that $|P_\varepsilon(x)| \leq C_0 Q_\varepsilon(x)$, $C_0 = \dfrac{\sup|(1/\Theta)_{zz}|}{\inf(1/\Theta)}$.

$$\Delta_x\rho_\varepsilon + Q_\varepsilon[m(x) - \rho_\varepsilon(x) + O(\varepsilon^2)] = 0 \text{ in } D, \quad n\cdot\nabla\rho_\varepsilon \text{ on } \partial D$$

$$\Rightarrow -O(\varepsilon^2) + \inf_\Omega m \leq \rho_\varepsilon(x) \leq \sup_\Omega m + O(\varepsilon^2)$$

$\Rightarrow \|\rho_\varepsilon\|_\infty \leq C$ and so is $|P_\varepsilon|$ and Q_ε.

Furthermore,

$$(1.1) \qquad |\Delta_x\rho_\varepsilon| \leq \varepsilon^2|P_\varepsilon| + Q_\varepsilon|m(x) - \rho_\varepsilon(x)|$$

RHS bounded in L^p $\forall p > 1$ $\Rightarrow \|\rho_\varepsilon\|_{W^{2,p}} \leq C(\|\Delta\rho_\varepsilon\|_\infty + \|\rho_\varepsilon\|_\infty)$.

In particular, Sobolev embedding implies #

$\rho_\varepsilon \to \rho$ in $C^1(\bar{\Omega})$ up to subsequence. $\varepsilon_k \to 0$.

Cor. $\exists$ at least one positive equilibrium sol.

Pf. Fix $\varepsilon > 0$. For $\tau \in [0,1]$

$$(P_\tau) \begin{cases} \mu(z)\Delta_x u + u\left(m(x) - (1-\tau)u - \tau\int_0^1 u\,dz\right) + \varepsilon^2 \partial_{zz} u = 0 & (x,z) \in D \times \mathbb{R} \\ n \cdot \nabla_x u = 0 & (x,z) \in \partial D \times \mathbb{R} \\ u(x,z) = u(x,z+1) & (x,z) \in D \times \mathbb{R}. \end{cases}$$

One can similarly find positive constant $C > 1$ which is independent of $\varepsilon > 0$ and $\tau \in [0,1]$ and any positive solution $\tilde{u}$ of (P_τ).

$$\frac{1}{C} \le \int_0^1 \tilde{u}(x,z)\,dz \le C$$

By Harnack principle, $\exists\, C'$ indep. of τ, $\tilde{u}$

s.t. $$\frac{1}{C'} \le \tilde{u}(x,z) \le C'$$

Now, when $\tau = 0$, (P_0) is the diffusive logistic equation and has a (unique) positive sol. One can then derive the existence of solution for (P_1) by a standard topological degree argument. See, e.g. [L. CVPDE (2007)].

Lemma 2. Suppose $\rho_\varepsilon \to \rho$ in $C^1(\bar{\Omega})$ for some $\rho \in C^1(\bar{\Omega})$, then $m - \rho \neq \text{const}$ in Ω.

Pf Integrate (1) we deduce $\int_D \rho_\varepsilon (m - \rho_\varepsilon)\, dx = 0$.

If $m - \rho_\varepsilon \to c_0$, then $c_0 = 0$.

But (1.1) implies $\Delta_x \rho_\varepsilon = o(1)$ in D, $n \cdot \nabla \rho_\varepsilon = 0$ on ∂D

$\Rightarrow \rho = \lim \rho_\varepsilon = \text{const.} \Rightarrow m - \rho \neq \text{const}$. #

Lemma 3 There exists C indep. of ε such that

$$\varepsilon \left\| \frac{\partial_z u_\varepsilon}{u_\varepsilon} \right\|_{C(\bar{D} \times [0,1])} + \left\| \frac{\nabla_x u_\varepsilon}{u_\varepsilon} \right\|_{C(\bar{D} \times [0,1])} \leq C$$

Pf. Fix $z_0 \in \mathbb{R}$ and define.

$$U_\varepsilon(x,y) = u_\varepsilon(x, z_0 + \varepsilon y)$$

Then $\mathcal{L}_\varepsilon U_\varepsilon = -A_\varepsilon(y) \Delta_x U_\varepsilon - \partial_{zz} U_\varepsilon = h_\varepsilon(x)\, U_\varepsilon$

where $A_\varepsilon(y)$ and $h_\varepsilon(x)$ are bdd Lipschitz (unif. in ε).

and $0 < \inf (H) \leq A_\varepsilon(y) \leq \sup (H)$.

Apply Harnack Principle,

(3.1) $$\sup_{D \times [-1,1]} U_\varepsilon \leq C \inf_{D \times [-1,1]} U_\varepsilon .$$

Sobolev embedding and L^p estimate implies

$$|\partial_z u_\varepsilon(x,0)| + |\nabla_x u_\varepsilon(x,0)| \leq C \|u_\varepsilon\|_{W^{2,p}(D\times[-\frac{1}{2},\frac{1}{2}])}$$
$$\leq C \|u_\varepsilon\|_{L^p(D\times[-1,1])}$$
$$\leq C \sup_{D\times[-1,1]} u_\varepsilon .$$

$$|\partial_z u_\varepsilon(x,0)| + |\nabla_x u_\varepsilon(x,0)| \leq C \inf_{D\times[-1,1]} u_\varepsilon \leq C\, u_\varepsilon(x,0) \quad \forall x\in D$$

#

Introduce the WKB-ansatz.

$$w_\varepsilon(x,z) = -\varepsilon \log u_\varepsilon(x,z)$$

then

$$(*) \quad -\varepsilon\, \partial_{zz} w_\varepsilon + |\partial_z w_\varepsilon|^2 + m(x) - f_\varepsilon(x) = \Theta(z)\left(\frac{\Delta_x w_\varepsilon}{\varepsilon} - \frac{|\nabla_x w_\varepsilon|^2}{\varepsilon^2}\right)$$

in $D\times\mathbb{R}$.

Only needed After Lemma 4

$$|\partial_z w_\varepsilon| = \varepsilon \frac{|\partial_z u_\varepsilon|}{u_\varepsilon}, \quad |\nabla_x w_\varepsilon| = \varepsilon \frac{|\nabla_x u_\varepsilon|}{u_\varepsilon}.$$

Lemma 4 $\{w_\varepsilon\}$ is precompact in $C(\bar{D}\times[0,1])$.

If $w_\varepsilon(x,z)\to w$, then $w=w(z)$ and $\inf_{[0,1]} w=0$.

Pf. By Lemma 3, $\sup_{D\times[0,1]}\left[|\partial_z w_\varepsilon| + \frac{1}{\varepsilon}|\nabla_x w_\varepsilon|\right] \le C$. equicontinuity.

Together with $\int_0^1 \exp\left(-\frac{w_\varepsilon(x,z)}{\varepsilon}\right)dz$

$$\int_0^1 \exp\left(-\frac{w_\varepsilon(x,z)}{\varepsilon}\right)dz = \rho_\varepsilon(x), \qquad 0<\frac{1}{2}\inf m \le \rho_\varepsilon(x) \le 2\sup m <+\infty,$$

We deduce by Ascoli theorem that

$$w_\varepsilon(x,z) \longrightarrow w(z) \quad \text{in } C(\bar{D}\times[0,1]) \text{ in a seq } \varepsilon_k\to 0.$$

where w is independent of x.

$\inf w>0 \Rightarrow \rho_\varepsilon\to 0$ impossible.

$\inf w<0 \Rightarrow \rho_\varepsilon\to +\infty$ impossible.

Hence, $\inf_{[0,1]} w = \lim_{\varepsilon\to 0}\inf_{D\times[0,1]} w_\varepsilon = 0$.

#.

Now, w_ε satisfies

$$(*)\quad -\varepsilon\partial_{zz}w_\varepsilon + |\partial_z w_\varepsilon|^2 + m(x) - \rho_\varepsilon(x) = \Theta(z)\left(\frac{\Delta_x w_\varepsilon}{\varepsilon} - \frac{|\nabla_x w_\varepsilon|^2}{\varepsilon^2}\right)$$

in $D\times\mathbb{R}$

Def For each $z \in \mathbb{R}$, and $h \in C(\bar{D})$, define

$\Lambda_h(z)$ and $x \mapsto \phi_h(x,z)$ to be the p.e.v. / p.e.f of

$$\begin{cases} \Theta(z)\,\Delta_x \phi = (m(x) - h(x))\phi + \Lambda\phi = 0 & \text{in } D \\ n\cdot\nabla_x\phi = 0 & \text{on } \partial D. \end{cases}$$

Recall $\Theta(z) > \Theta(\hat{z}) \quad \forall z \in [0,1],\ z \neq \hat{z}$.

Suppose $W_\varepsilon(x,z) \to W(z)$ and $\rho_\varepsilon(x) \to \rho(x)$ uniformly.

Define $\psi(x,z) = \log \phi_\rho(x,z)$, then

$$-\Theta(z)\left(|\nabla_x\psi|^2 + \Delta_x\psi\right) = m(x) - \rho(x) + \Lambda_\rho(z).$$

Write $W_\varepsilon(x,z) := V_\varepsilon(x,z) - \varepsilon\psi(x,z)$,

$$\underset{+o(1)}{-\varepsilon\partial_{xx}V_\varepsilon} + |\partial_z V_\varepsilon + o(1)| + m(x) - \rho_\varepsilon = \Theta(z)\left(\frac{\Delta_x V_\varepsilon}{\varepsilon} - \Delta_x\psi - \frac{|\nabla_x V_\varepsilon|^2}{\varepsilon^2} + 2\frac{\nabla_x V_\varepsilon \nabla_x\psi}{\varepsilon} - |\nabla_x\psi|^2\right)$$

$$= \Theta(z)\left(\frac{\Delta_x V_\varepsilon}{\varepsilon} + 2\frac{\nabla_x V_\varepsilon \nabla_x\psi}{\varepsilon} - \frac{|\nabla_x V_\varepsilon|^2}{\varepsilon}\right) + m(x) - \rho(x) + \Lambda_\rho(z).$$

$$(*)\quad -\varepsilon\partial_{xx}V_\varepsilon + |\partial_z V_\varepsilon + o(1)|^2 = \Theta(z)\left(\frac{\Delta_x V_\varepsilon}{\varepsilon} - \frac{2\nabla_x V_\varepsilon\nabla_x\psi}{\varepsilon} - \frac{|\nabla_x V_\varepsilon|^2}{\varepsilon^2}\right) + \Lambda_\rho(z) + o(1).$$

(Keep on board).

Lemma 5 $\quad \Lambda_\rho(z) > 0 \quad$ if $\quad z \in [0,1] \setminus \{\hat{z}\}$

Pf. $\quad V_\varepsilon(x,z) = W_\varepsilon(x,z) + o(1) \to W(z)$.

Fix an interval $[p,q] \subseteq \mathbb{R}$, take a test fcn

$\varphi(z) = \dfrac{1}{\sigma(z)}, \quad \sigma(z) > 0$ in (p,q), $\sigma(z) \equiv 0$ in $\mathbb{R} \setminus (p,q)$.

then $\quad V_\varepsilon(x,z) - \varphi(z)$ attains maximum point in $\bar{\Omega} \times [p,q]$.
$(x_\varepsilon, z_\varepsilon)$.

WLOG, $z_\varepsilon \to z_0 \in [p,q]$.

$(*) \Rightarrow \quad |\varphi'(z_\varepsilon) + o(1)| + o(1) \leq \Lambda_\rho(z_\varepsilon)$

Letting $\varepsilon \to 0$, $\quad 0 \leq \Lambda_\rho(z_0) \quad \Rightarrow \Lambda_\rho \geq 0$ in a dense set.

By continuity, $\Lambda_\rho(z) \geq 0 \quad \forall z \in \mathbb{R}$.

Since $m(x) - \rho(x) \neq$ const, Λ_ρ is strictly increasing in diffusion rate.

$\forall z \neq \hat{z}$

$\Theta(z) > \Theta(\hat{z}) \quad \Rightarrow \quad \Lambda_\rho(z) > \Lambda_\rho(\hat{z}) \geq 0$.

#

Lemma 6 $\quad w(z) > 0 \quad$ for $\quad z \in [0,1] \setminus \{\hat{z}\}$.

Pf. Suppose $w(z_0) = 0 \quad \exists z_0 \in [0,1]$,

Then $\quad w(z) + (z - z_0)^2$ has a strict local min at $z = z_0$.

$\Rightarrow \quad V_\varepsilon(x,z) + (z - z_0)^2$ has a local min $(x_\varepsilon, z_\varepsilon) \in \bar{D} \times \mathbb{R}$

s.t. $z_\varepsilon \to z_0$.

At the point $(x_\varepsilon, z_\varepsilon)$,

$$\varepsilon \left[(z - z_0)^2 \right]''_{z = z_\varepsilon} + \left| 2(z_\varepsilon - z_0) + o(1) \right|^2 \geq \Lambda_p(z_\varepsilon) + o(1).$$

Letting $\varepsilon \to 0$, $\quad 0 \geq \Lambda_p(z_0). \Rightarrow z_0 = \hat{z}$.

To conclude, for $z \neq \hat{z}$, $w(z) > 0$ and,

$$U_\varepsilon(x,z) = \exp\left(-\frac{w(z)+o(1)}{\varepsilon}\right) \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0.$$

and we claim that $\rho(x) = \lim_{\varepsilon\to 0} \rho_\varepsilon(x)$ is a weak solution of
$$\begin{cases} -\Theta(\hat{z})\,\Delta_x \rho = \rho(m-\rho) & \text{in } \Omega \\ n\cdot\nabla_x \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Indeed, rewrite (0,1) as

$$\Theta(\hat{z})\,\Delta_x U_\varepsilon + U_\varepsilon(m(x)-\rho_\varepsilon(x)) = -\left(\Theta(z)-\Theta(\hat{z})\right)\Delta_x U_\varepsilon - \varepsilon^2 \partial_{zz} U_\varepsilon$$

Integrate in z.

$$\Theta(\hat{z})\,\Delta \rho_\varepsilon + \rho_\varepsilon(m-\rho_\varepsilon) = -\int_0^1 \left(\Theta(z)-\Theta(\hat{z})\right)\Delta_x U_\varepsilon\, dz.$$

Mult. by a test fcn $\varphi(x) \in C^1(\bar{D})$,

$$-\Theta(\hat{z})\int_D \nabla\rho_\varepsilon \nabla\varphi + \int_D \varphi\rho_\varepsilon(m-\rho_\varepsilon) = R_\varepsilon.$$

where

$$|\tilde{R}| = \left|\iint_{D\times[0,1]} \left(\Theta(z)-\Theta(\hat{z})\right)\nabla_x U_\varepsilon\cdot\nabla_x\varphi\, dx\,dz\right|$$

$$\leq C\int_{D\times[0,1]} |z-\hat{z}|\, U_\varepsilon(x,z)\, dz\,dx \leq C\left(\int_{D\times[\hat{z}-\delta,\hat{z}+\delta]} + \int_{D\times[\hat{z}-\delta,\hat{z}+\delta]^c}\right)$$

$$\leq C\left(\delta\int_D \rho_\varepsilon dx + o(1)\right).$$

$$\Rightarrow \limsup_{\varepsilon\to 0}|R_\varepsilon| \leq C\delta \quad \underline{\forall \delta > 0} \Rightarrow |R_\varepsilon| \to 0 \text{ as } \varepsilon\to 0.$$

Therefore,

$$\textcircled{H}(\hat{z})\int_D \nabla\rho\nabla\varphi\,dx + \int_D \varphi\rho(m-\rho)dx = 0 \quad \forall\varphi\in C^1(\bar{D}).$$

i.e. ρ is a weak solution to

$$\begin{cases} \textcircled{H}(\hat{z})\Delta\rho + \rho(m-\rho) = 0 & \text{in } \Omega \\ n\cdot\nabla\rho = 0 & \text{on } \partial\Omega \end{cases}$$

By uniqueness, $\rho = \theta_{\hat{z}}$. #.

Remark. In fact, the limits $w(z) = \lim_{\varepsilon\to0} w_\varepsilon(x,z)$
$\rho(x) = \lim_{\varepsilon\to0}\rho_\varepsilon(x)$

satisfies a constraind Hamilton-Jacobi equation

$$\begin{cases} |w'|^2 = \Lambda_\rho(z) & \text{for } 0<z<1 \text{ in viscosity sense} \\ \inf_{0<z<1} w(z) = 0 \end{cases}$$

To determine the "parameter" $\rho(x)$ who lives in $C(\bar{D})$, we use the reduction principle to deduce that
$\Lambda_\rho(z) > 0$ for $z \neq \hat{z}$ $\quad (\hat{z} = \arg\min \textcircled{H})$.
By property of viscosity sol, w can only attain minimum at $z = \hat{z}$ $\Longrightarrow$ $w(z) > 0$ for $z \neq \hat{z}$

$$\Longrightarrow u_\varepsilon(x,z) = \exp\left(-\frac{w(z)+o(1)}{\varepsilon}\right) \approx \rho(x)\,\delta(z-\hat{z})$$

Hence, ρ can be uniquely determined by the single species model.

Questions of Uniqueness & Stability of Equilibrium

Case 1 $m(x) \not\equiv \text{const}$.

In a slight variant of the problem (where $\Theta(z) = z$ and the periodic b.c. in z variable is substituted by a Neumann b.c.), it is proved that the positive equilibrium is locally asymptotically stable and unique when $0 < \varepsilon \ll 1$. The main idea is to show that every positive equilibrium is linearly stable, and use a topological degree argument.

Case 2 $m(x) \equiv 1$.

Consider time-dependent solutions $u(x, z, t)$ of

$$\begin{cases} \partial_t u - \varepsilon^2 \partial_{zz} u = \Theta(z) \Delta_x u + u(1 - \rho(x,t)) & x \in D,\ z \in \mathbb{R},\ t > 0 \\ \text{Neumann b.c. in } x,\ 1\text{-periodic in } z. \\ \rho(x,t) = \int_0^1 u(x,z,t)\, dz. \end{cases}$$

- $U(x, a) \equiv 1$ is a positive equilibrium.
- $V(u) := \int_D \int_0^1 (u - 1 - \log u)\, dz\, dx$ is a Lyapunov function.

$$\frac{d}{dt} V(u) = \iint \partial_t u - \frac{\partial_t u}{u} \, dz\, dx$$

$$= \iint u(1-\rho) - \frac{\varepsilon^2 \partial_{zz} u + \Theta \Delta_x u}{u} - (1-\rho) \, dz\, dx$$

$$= \iint (u-1)(1-\rho)\, dz\, dx - \iint \frac{\varepsilon^2 |\partial_z u|^2 + \Theta |\partial_x u|^2}{u^2} \, dz\, dx$$

$$\leq - \int_D (1-\rho)^2 \, dx .$$

By LaSalle's invariance principle,

$$u(\cdot, \cdot, t) \longrightarrow \text{maximal invariance set of } \left\{ \begin{array}{l} \text{entire solutions} \\ \hat{u}(x,z,t) \text{ for which} \end{array} \quad \int_0^1 \hat{u}(x,z,t)\, dz \equiv 1 \right\}$$

i.e. $\hat{u}(x,z,t)$ satisfies

$$\begin{cases} \partial_t \hat{u} = \varepsilon^2 \partial_{zz} \hat{u} + \Theta(z) \Delta_x \hat{u} & \text{in } D \times \mathbb{R} \times \mathbb{R} \\ n \cdot \nabla_x \hat{u} = 0 & \partial D \times \mathbb{R} \times \mathbb{R} \\ \hat{u}(x,z,t) = \hat{u}(x, z+1, t), \quad \hat{u} > 0 & D \times \mathbb{R} \times \mathbb{R} \end{cases}$$

By uniqueness of principal Floquet bundle,

$\hat{u}(x,z,t) \equiv C \qquad \exists\, C > 0.$

Since $\int_0^1 \hat{u}\, dz \equiv 1$, we deduce that $\hat{u} \equiv 1$.

<u>Conjecture</u>, Global stability of equilibrium holds in general. #

Selection-Mutation Model in advective environment.

Consider the equilibrium solutions $u_\varepsilon(x,z)$ of

$$\begin{cases} \partial_t u - \varepsilon^2 \partial_{zz} u = z\partial_{xx} u - q\partial_x u + u\left(m(x) - \rho_\varepsilon(x,t)\right) & 0<x<L,\ z\in I,\ t>0 \\ z\partial_x u - qu = 0 & x=0,L,\ z\in I,\ t>0. \\ u=0 & 0\le x\le L,\ z\in\partial I,\ t>0 \\ u(x,z,0)=u_0(x,z). \end{cases}$$

where $I = [\underline{z}, \overline{z}] \subseteq (0,\infty)$.

Then under smallness assumptions on $|I|$, one can obtain:

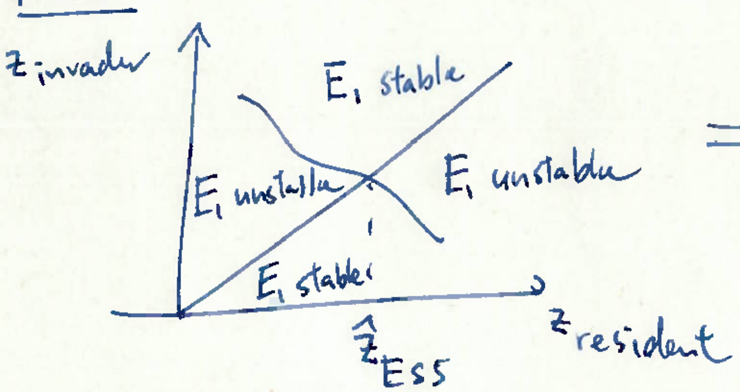
(ESS) If the adaptive dynamics of two-species model has a ESS $\hat{z}\in I$, then for $0<\varepsilon<<1$,

$$u_\varepsilon(x,z) \longrightarrow \theta_{\hat{z}}(x)\,\delta(z-\hat{z}).$$

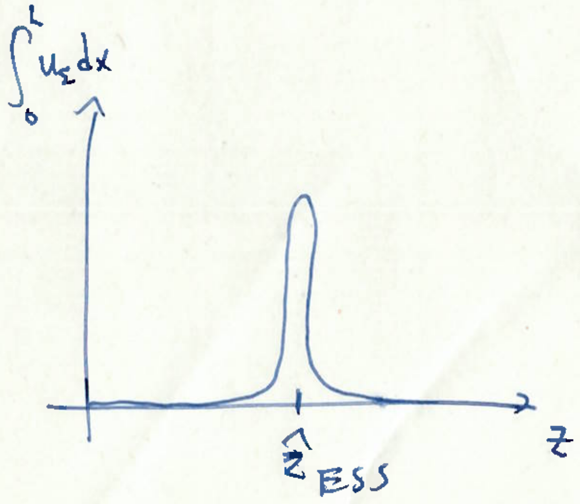
(Fixation) If the AD says slower diffusion (resp. faster diffusion) wins, then

$$u_\varepsilon(x,z) \to \theta_{\underline{z}}(x)\,\delta(z-\underline{z}) \qquad \left(\text{resp. } \theta_{\overline{z}}(x)\,\delta(z-\overline{z})\right).$$

<u>Recall</u>

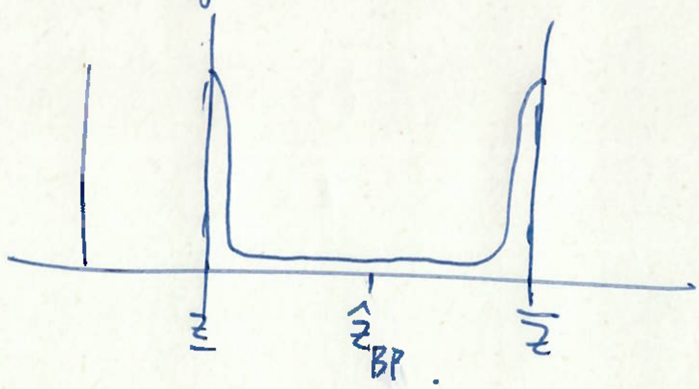


$\Longrightarrow$



(Branching). If the AD has a branching point $\hat{z} \in I$,

then
$$u_\varepsilon(x,z) \longrightarrow U(x)\,\delta(z-\underline{z}) + V(x)\,\delta(z-\overline{z})$$
where $(U(x), V(x))$ is a positive equilibrium of the underlying two-species competition model.



Lorz - Mirrahimi - Perthame. (2011).

$$\begin{cases} \varepsilon \partial_t u_\varepsilon - \varepsilon^2 \Delta u_\varepsilon = u_\varepsilon \big(m(z) - \rho_\varepsilon(t)\big) & \text{for } z \in \mathbb{R}^1, t>0 \\ \rho_\varepsilon(t) = \int u_\varepsilon(z,t)\,dz . \\ u_\varepsilon(z,0) = \exp\left(-\frac{|z-z_0|^2 + o(1)}{\varepsilon}\right) \approx \delta(z-z_0) \end{cases}$$

Take
$$W_\varepsilon(x,z,t) = -\varepsilon \log u_\varepsilon(x,z,t)$$
then

$$\begin{cases} \partial_t W_\varepsilon - \varepsilon \Delta W_\varepsilon + |\partial_z W_\varepsilon|^2 + m(z) - \rho_\varepsilon(t) = 0 & z \in \mathbb{R}^n, t>0 \\ W_\varepsilon(z,0) = W_\varepsilon^0(z) \approx (z-z_0)^2 . \end{cases}$$

By establishing appropriate estimates, it can be shown that $W_\varepsilon(z,t) \longrightarrow w(z,t)$ in $C_{loc}(\mathbb{R}^n \times \mathbb{R}_+)$ and $\exists\, \rho \in BV(\mathbb{R}_+)$ s.t. $\rho_\varepsilon(t) \longrightarrow \rho(t)$ in $L^1_{loc}(\mathbb{R}_+)$.

s.t.
$$\begin{cases} \partial_t w + |\partial_z w|^2 + m(z) - \rho(t) = 0 & \text{in } \mathbb{R}^1 \times \mathbb{R}_+ \text{ in viscosity sense} \\ \inf_{\mathbb{R}^n} w(\cdot, t) = 0 & \forall t > 0 . \end{cases}$$

$$n_\varepsilon(z,t) \xrightarrow[\varepsilon \to 0]{} \rho(t)\, \delta(z - \bar{z}(t)) \quad \text{s.t.} \quad m(\bar{z}(t)) = \rho(t) .$$

Canonical equation of Adaptive Dynamics

$$\frac{d}{dt}\bar{z}(t) = \left(-D^2 u(\bar{z}(t),t)\right)^{-1} \cdot \nabla m(\bar{z}(t)) , \quad \bar{z}(0) = z_0 .$$

§ Time-dependent problem

Difficulty : There is no reduction principle, how do we determine the parameter $\rho_\varepsilon(x,t)$??

$$\varepsilon \partial_t \rho_\varepsilon - \Delta_x \int (14)(z) u_\varepsilon \, dz = \rho_\varepsilon (m - \rho_\varepsilon) \quad \text{in } D \times (0,\infty)$$

Under additional assumption on (14) ← mutation distance. metric in the phenotypic space,

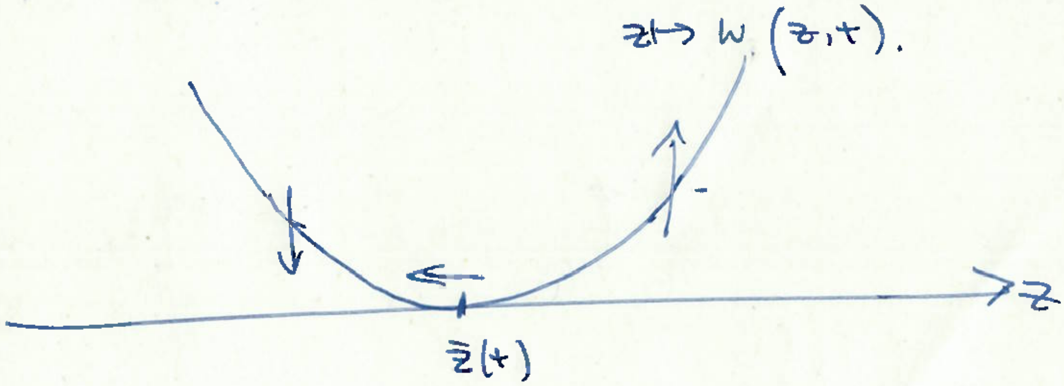
one can show that $w(z,t)$ retains convexity in z,

so one can show that (for specially chosen initial condition).

$$u_\varepsilon(x,z,t) = \delta(z - \bar{z}(t)) \, \theta_{\bar{z}(t)}(x)\,,$$

where $\bar{z}(t) \in C^1_{loc}$ is the "unique" path s.t. certain Hamilton-Jacobi eq has a viscosity sol.

$$\begin{cases} \partial_t w + |\partial_z w|^2 = \lambda(z, \bar{z}(t)) & \text{for } z \in \mathbb{R},\ t>0 \\ \inf_z w(\cdot,t) = 0 & \text{for } t>0 \\ \lambda(z,\bar{z}) = \Lambda_{\theta_{\bar{z}}}(z) & \end{cases}$$



For the case $\textcircled{H}(z) = z$, $z \in [a,b] \subseteq (0,\infty)$, it seems that one will need the uniqueness result in [Calvez-L. (CPDE (2018)] and also the hypothesis that Dockery's conjecture holds for all $N \geq 3$ and all $0 < z_1 < \dots < z_n$, to handle