# Asymptotic Spreading of the Fisher-KPP Equation

We discuss the F-KPP model in 1-dim, unbounded domain,

$$\text{(FKPP)} \quad \begin{cases} u_t - u_{xx} = u(r(x,t) - u) & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

where $0 \le u_0(x) \le 1$, $u \not\equiv 0$, and $u_0(x) = 0$ for all sufficiently large $x$.

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**Def.** We say that (F-KPP) has (right) spreading speed $c_* > 0$ if

(a) $c > c_* \implies \lim_{t\to\infty} \sup_{x > ct} |u(x,t)| = 0$

(b) $0 < c < c_* \implies \liminf_{t\to\infty} \inf_{0<x<ct} u(x,t) > 0$ (in case $\inf r > 0$)

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We will restrict our discussion to the following two cases:

(Homogeneous environment) $r(x,t) \equiv r_0$

(Shifting environment) $r(x,t) = g(x - ct)$

## § Homogeneous environment $r(x,t) = r_0 > 0$

### Traveling wave solutions

(F-KPP) admits a family of TW sol. $\{u_c(x,t) : c \geq 2\sqrt{r_0}\}$ such that $u_c(x,t) = p(x-ct)$, where $p(\xi)$ and satisfies

$$\begin{cases} -cp' - p'' = p(r_0 - p) & \xi \in \mathbb{R}. \\ p(-\infty) = r_0, \quad p(+\infty) = 0. \end{cases}$$

In particular, $2\sqrt{r_0}$ is the minimal speed of TW sol.

Pf. Phase-plane method:

$$\begin{cases} p' = q \\ q' = -cq - q(r_0 - q) \end{cases}$$

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We will show that the spreading speed $c_*$ coincides with the minimal wave speed $2\sqrt{r_0}$.

Thm A [Kolmogorov et al. (1937); Aronson-Weinberger (1978)].

$$r(x,t) \equiv r_0 > 0 \implies c_* = 2\sqrt{r_0}.$$

Remark The coincidence of $c_*$ with the minimal TW speed holds in much greater generality.
See [Weinberger (1982); Liang-Zhao (2007)].

$c_* t + o(t)$

Convergence to TW profile

$$\sup_{x>0} \left| u(x,t) - \phi\left(x - 2\sqrt{r_0}\,t + \frac{3}{2\sqrt{r_0}} \log t + C_0\right) \right| \longrightarrow 0 \quad \text{as } t \to \infty$$

[Bramson, Mem. AMS (1983)]

[Hamel-Nolen-Roquejoffre-Ryzhik, Net. Het. Media (2013)]

[Nolen-Roquejoffre-Ryzhik, Chin. Ann. Math. (2017)]

We give a proof of Thm A by constructing super/subsolutions in the generalized sense.

**Supersol** For $r_0 > 0$, $\lambda > 0$, define

$$\phi_{r_0,\lambda}(x,t) = e^{-\lambda(x-ct)}, \quad \text{where } c = \lambda + \frac{r_0}{\lambda}.$$

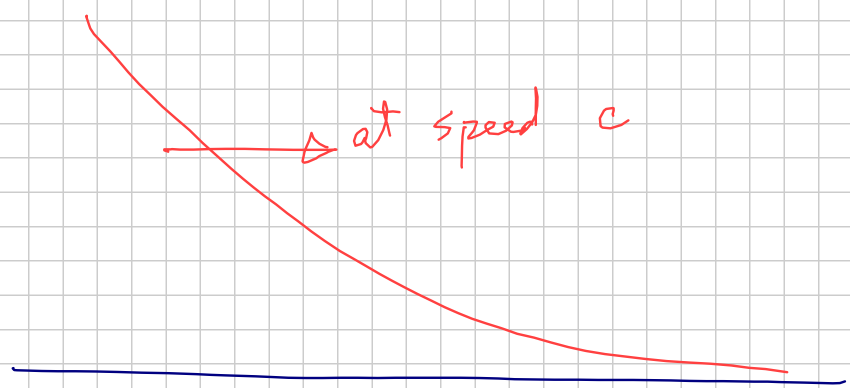
Then $\phi_t - \phi_{xx} - \phi(r_0 - \phi) \geq 0 \quad \forall x \in \mathbb{R},\ t \geq 0.$

Indeed, $\phi_t - \phi_{xx} - \phi(r_0-\phi)$

$$\geq \phi_t - \phi_{xx} - r_0\phi$$

$$= \phi\left[\lambda c - \lambda^2 - r_0\right]$$

$$= 0.$$



**Subsol.** For $\alpha \geq 0$, $c \geq 0$, $r_0 > 0$ and $R > 0$, define

$$\psi(x,t) = \psi_{\alpha,c,r_0,R}(x,t) = \begin{cases} \exp\left(-\alpha t - \frac{c}{2}(x-ct)\right) \sin\left(\frac{x-ct}{R}\right), & 0 < \frac{x-ct}{R} < \pi \\ 0 & \text{otherwise.} \end{cases}$$


$\alpha > -r_0 + \frac{c^2}{4}$

**Claim** For $\eta > 0$ such that $\alpha \geq -r_0 + \frac{c^2}{4} + \frac{1}{R^2} + \eta$,

$$(\eta\psi)_t - (\eta\psi)_{xx} - (\eta\psi)(r_0 - \eta\psi) \leq 0 \quad \text{in } \mathbb{R} \times (0,\infty) \text{ in the generalized sense.}$$

**Check** In $\left\{0 < \frac{x-ct}{R} < \pi\right\}$,

$$\psi_t - \psi_{xx} - r_0\psi = -(\alpha + r_0)\psi + e^{-\alpha t - \frac{c}{2}(x-ct)} \left\{ \overbrace{\left[\frac{c^2}{2}\sin(\ ) - \cancel{\frac{c}{R}\cos(\ )}\right]}^{\partial_t} - \overbrace{\left[\frac{c^2}{4}\sin(\ ) - \cancel{\frac{c}{R}\cos(\ )} - \frac{1}{R^2}\sin(\ )\right]}^{\partial_{xx}} \right\}$$

$$= -(\alpha + r_0)\psi + \psi\left\{\frac{c^2}{2} - \frac{c^2}{4} - \frac{1}{R^2}\right\} = \psi\left[-\alpha - r_0 + \frac{c^2}{4} + \frac{1}{R^2}\right].$$

$$(\eta\psi)_t - (\eta\psi)_{xx} - \eta\psi(r_0 - \eta\psi) = \eta\psi\left[-\alpha - r_0 + \frac{c^2}{4} + \frac{1}{R^2} + \eta\cancel{\psi}\right] \leq 0$$

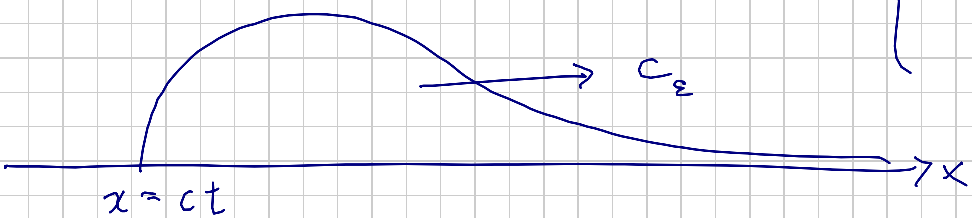
$$\boxed{0 \leq \psi \leq 1.}$$

<u>Subsol</u> Given $\lambda \in (0, \sqrt{r_0})$ and any $\varepsilon > 0$ suff small such that

$$\lambda < \sqrt{r_0 - \varepsilon} \quad \text{and} \quad 0 < \varepsilon < \frac{r_0 - \varepsilon}{\lambda} - \lambda$$

Let $c = c_\varepsilon = \lambda + \dfrac{r_0 - \varepsilon}{\lambda}$, then $\varepsilon < c - 2\lambda$.

Define $\rho(x,t) = \rho_{r_0,\lambda,\varepsilon}(x,t) = \begin{cases} e^{-\lambda(x-ct)} - e^{-(\lambda+\varepsilon)(x-ct)} & \text{for } x - ct > 0 \\ 0 & \text{for } x - ct \le 0. \end{cases}$




Note that $\sup \rho \le 1$.

<u>Claim</u> For $0 < \eta \le \varepsilon$, $(\eta\rho)_t - (\eta\rho)_{xx} - (\eta\rho)(r_0 - \eta\rho) \le 0$, in gen. sense.

<u>Check</u> $\rho_t - \rho_{xx} - (r_0 - \varepsilon)\rho$

$$= e^{-\lambda(x-ct)}\left[c\lambda - \lambda^2 - r_0 + \varepsilon\right] - e^{-(\lambda+\varepsilon)(x-ct)}\left[c(\lambda+\varepsilon) - (\lambda+\varepsilon)^2 - r_0 + \varepsilon\right]$$

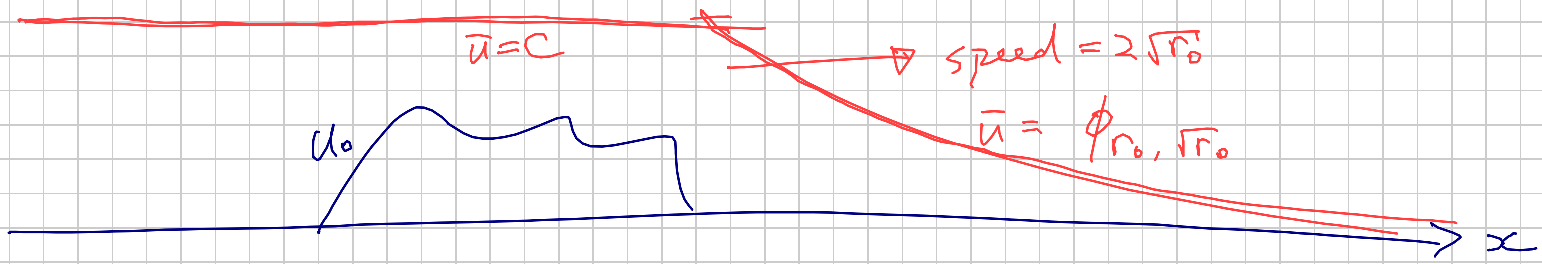
$$= -e^{-(\lambda+\varepsilon)(x-ct)}\left[c\varepsilon - 2\lambda\varepsilon - \varepsilon^2\right] = -e^{-(\lambda+\varepsilon)(x-ct)}\varepsilon\left[c - 2\lambda - \varepsilon\right] \le 0$$

$$(\eta\rho)_t - (\eta\rho)_{xx} - \eta\rho(r_0 - \eta\rho) \le \eta\rho(-\varepsilon + \eta\rho) \le \eta\rho(-\varepsilon + \eta) \le 0.$$

Pf of Thm A (a) : $c > 2\sqrt{r_0} \Rightarrow \lim\limits_{t\to\infty} \sup\limits_{x>ct} |u(x,t)| = 0$.

Consider the generalized supersolution $\bar{u}(x,t) = \min\left\{ \sup\limits_{\mathbb{R}} u_0 \,,\; e^{-\sqrt{r_0}(x - 2\sqrt{r_0}\,t)} \right\}$

WLOG, we may translate $u_0$ to ensure $0 \le u_0(x) \le \bar{u}(x,0)$




$$\Longrightarrow \quad 0 \le u(x,t) \le \bar{u}(x,t) \qquad \forall x \in \mathbb{R},\ t > 0$$

For $c > 2\sqrt{r_0}$,

$$\lim_{t\to\infty} \sup_{x>ct} u(x,t) \le \lim_{t\to\infty} \sup_{x>ct} \bar{u}(x,t) \le \lim_{t\to\infty} e^{-\sqrt{r_0}(c - 2\sqrt{r_0})t} = 0$$

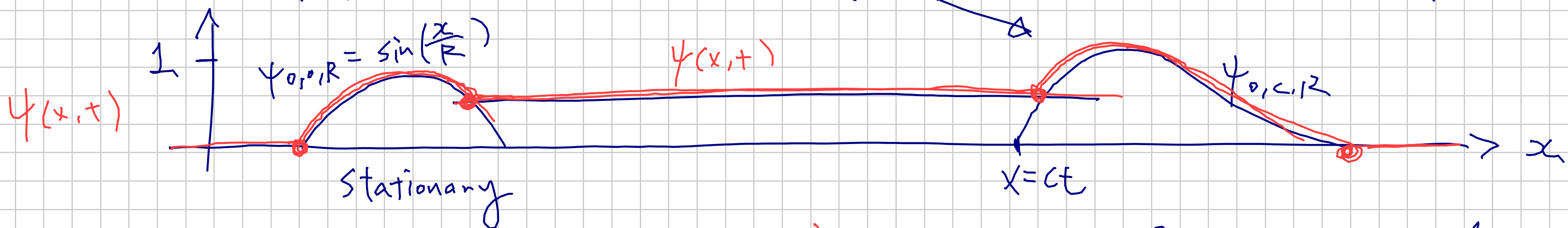
This proves (a).

Pf of Thm A (b) : $0 < c < 2\sqrt{r_0} \Rightarrow \liminf_{t\to\infty} \inf_{0<x<ct} u(x,t) > 0$ ..

Since $0 < c < 2\sqrt{r_0}$, we may choose $R$ large and $\eta$ small, (and $\alpha = 0$)

so that $0 > \underbrace{-r_0 + \frac{c^2}{4}} + \frac{1}{R^2} + \eta$ and $0 > -r_0 + \frac{1}{R^2} + \eta$.

$\Rightarrow \eta\psi_{0,0,R}$ and $\eta\psi_{0,c,R}$ are subsol. $\forall \eta$ small




$\Rightarrow \eta\psi(x,t)$ ($\psi$ defined in red) is subsol for $\eta > 0$ small.

Pf of Thm A (b): $0<c<2\sqrt{r_0} \Rightarrow \liminf_{t\to\infty} \inf_{0<x<ct} u(x,t) > 0$

Since $0<c<2\sqrt{r_0}$, we may choose $R$ large and $\eta$ small, (and $\alpha=0$)

so that $0 > -r_0 + \frac{c^2}{4} + \frac{1}{R^2}$ and $0 > -r_0 + \frac{1}{R^2}$.

$\Rightarrow$ $\eta\,\psi_{0,0,R}$ and $\eta\,\psi_{0,c,R}$ are subsol. $\forall \eta$ small


$\Rightarrow$ $\eta\,\psi(x,t)$ ($\psi$ defined in red) is subsol for $\eta>0$ small.

Given a nonnegative, nontrivial sol $u(x,t)$ of (F-KPP).

Strong max. prin. $\Rightarrow$ $u(x,1)>0$ $\forall x\in\mathbb{R}$.

Choose $\eta$ small enough s.t. $u(x,1) \geq \eta\,\psi(x,1)$ $\forall x\in\mathbb{R}$.

Comparison principle $\Rightarrow$ $u(x,t) \geq \eta\,\psi(x,t)$ $\forall x\in\mathbb{R}$, $\forall t\geq 1$

This proves (b).

## Spatially perodic environments

$$\begin{cases} u_t - u_{xx} = u(r(x)-u), \\ u(x,0) \text{ compactly supported.} \end{cases}$$

If $r(x) \equiv r_0$, then $\tilde{u} = e^{-\lambda x + \mu t}$ $(\mu = \lambda^2 + r_0)$ solves

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} = r_0 \tilde{u} & x \in \mathbb{R},\ t \in \mathbb{R} \\ \text{for each } t,\ \tilde{u}(x,t) \sim e^{-\lambda x} \text{ for } x \gg 1. \end{cases}$$

$v = e^{\mu t} = e^{\lambda x} \tilde{u}$ solves

$$\begin{cases} v_t - v_{xx} + 2\lambda v_x = (\lambda^2 + r_0) v & x \in \mathbb{R},\ t \in \mathbb{R} \\ v(\cdot, t) \in L^\infty(\mathbb{R}) & t \in \mathbb{R} \end{cases}$$

Note that $\lambda^2 + r_0$ is the p.e.v. and $c^* = \inf_{\lambda > 0} \dfrac{\lambda^2 + r_0}{\lambda}$.

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If $r(x)$ is periodic in $x$, one use $e^{-\lambda x + \mu t} \psi(x)$, $= e^{-\lambda (x - \frac{\mu(\lambda)}{\lambda} t)} \psi(x)$

where $\mu = \mu(\lambda)$ and $\psi(x)$ is the p.e.v. and p.e.f of

$$\begin{cases} -\psi_{xx} + 2\lambda \psi_x = (\lambda^2 + r(x)) \psi + \mu \psi & \text{in } \mathbb{R} \\ \psi \text{ is periodic in } x. \end{cases}$$

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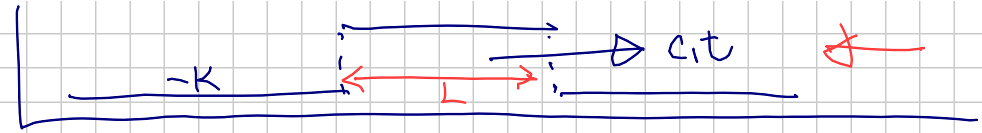
This suggests that $c_* = \inf_{\lambda > 0} \dfrac{\mu(\lambda)}{\lambda}$.

[Freidlin Gärtner (1979); Weinberger (2002); Berestycki - Hamel - Nadin (2008)]

# § Shifting Environments




$$\begin{cases} u_t - u_{xx} = u\,(g(x-c_1 t) - u) & x\in\mathbb{R},\ t>0 \\ u(x,0) = u_0(x) \text{ is compactly supported in } \mathbb{R}. \end{cases}$$

Motivated by the poleward moment of isotherms [Potapov-Lewis (2004)] and [Berestycki-Diekmann-Nagelkerke-Zegeling (2009)] considered the case $r(x,t) - u = g(x-ct) - u = \begin{cases} 1-u & c_1 t < x < c_1 t + L \\ -K & \text{otherwise}. \end{cases}$

Changing variable $y = x - c_1 t$, we obtain

$$\tilde{u}_t - \tilde{u}_{yy} - c_1 \tilde{u}_y = \begin{cases} \hat{u}(1-\tilde{u}) & 0<y<L,\ t>0 \\ -K\tilde{u} & y\notin[0,L],\ t>0. \end{cases}$$

The long-time dynamics is equivalent to the bounded domain problem

$$\begin{cases} U_t - U_{yy} - c_1 U_y = U(1-U) & y\in[0,L],\ t>0 \\ U_y - k_+ U = 0 & y=0,\ t>0 \\ U_y - k_- U = 0 & y=L,\ t>0 \end{cases}$$

where $k_\pm = \dfrac{-c_1 \pm \sqrt{c_1^2+4K}}{2}$

The long-time dynamics is equivalent to the bounded domain problem

$$\begin{cases} u_t - u_{yy} - c_1 u_y = u(1-u) & y \in [0,L],\ t>0 \\ u_y - k_+ u = 0 & y=0,\ t>0 \\ u_y - k_- u = 0 & y=L,\ t>0 \end{cases}$$

where $k_\pm = \dfrac{-c_1 \pm \sqrt{c_1^2+4k}}{2}$

By lecture 1, the population can persist on the moving patch if and only if $\mu_1 < 0$, where $\mu_1 = \mu_1(c_1, L, k)$ is the principal eigenvalue of

$$\begin{cases} -\psi'' - c_1\psi' = \psi + \mu_1 \psi & 0<y<L \\ \psi'(0) - k_+\psi(0) = \psi'(L) - k_-\psi(L) = 0. \end{cases}$$

<u>Proposition</u> (Berestycki et al.) For each $0 < c_1 < 2$ and $k>0$, there exists $L_{crit} > 0$ such that the population persists iff

$$L > L_{crit} := \frac{1}{\sqrt{1-\left(\frac{c_1}{2}\right)^2}} \arctan\left( \frac{2\sqrt{1-\left(\frac{c_1}{2}\right)^2}\sqrt{\frac{1}{k}+\left(\frac{c_1}{2}\right)^2}}{1-k-c_1^2/2} \right).$$

Another situation was considered in [Li-Bewick-Shang-Fagan (2014)]

**Thm B** $\quad r(x,t) = g(x - c_1 t)$, $\quad g$ monotone increasing, $\quad \underset{r_1}{\underset{\parallel}{g(-\infty)}} < 0 < \underset{r_2}{\underset{\parallel}{g(+\infty)}}$

(a) $\quad c_1 > 2\sqrt{r_2} \implies$ (extinction) $\quad \lim\limits_{t\to\infty} \sup\limits_{x\in\mathbb{R}} u(x,t) = 0$.

(b) $\quad 0 < c_1 < 2\sqrt{r_2} \implies$ (persistence + spreading at $2\sqrt{r_2}$)

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{x > (2\sqrt{r_2}+\varepsilon)t} u(x,t) = 0 & \forall\, \varepsilon > 0 \\ \liminf\limits_{t\to\infty} \inf\limits_{(c_1+\varepsilon)t < x < (2\sqrt{r_2}-\varepsilon)t} u(x,t) > 0 & \forall\, 0 < \varepsilon \ll 1. \end{cases}$$

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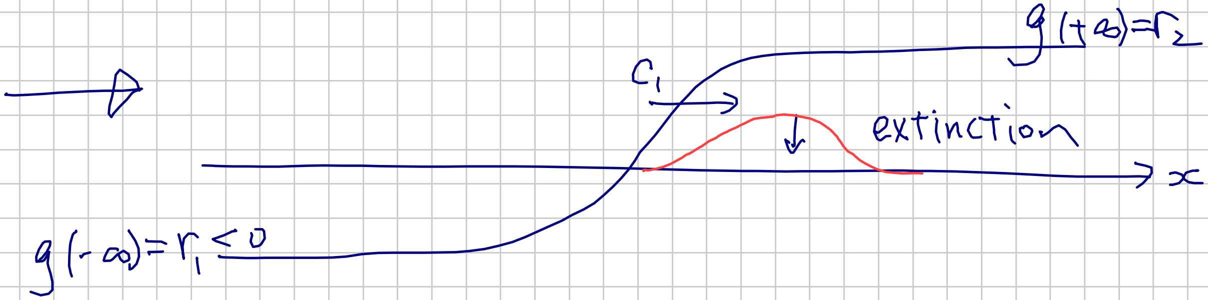
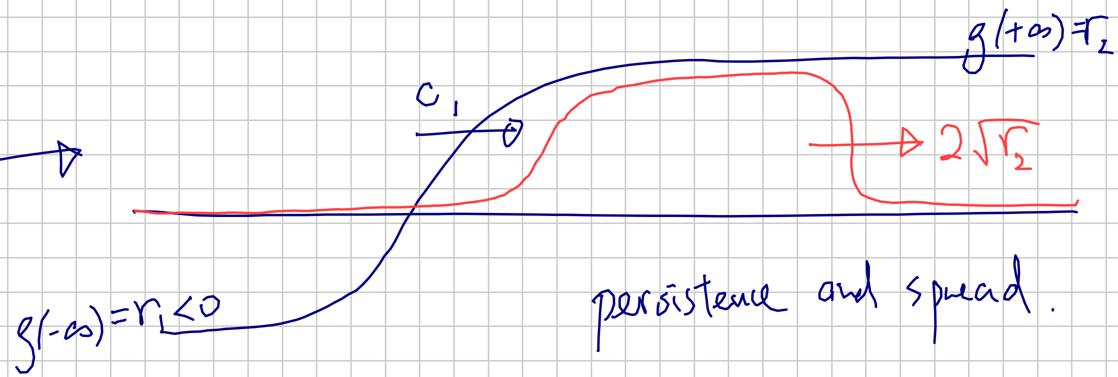
$c_1 > 2\sqrt{r_2}$ (shifting too fast) $\longrightarrow$




$0 < C < 2\sqrt{r_2}$ (shifting is moderate (slow)) $\longrightarrow$




persistence and spread.

Another situation was considered in [Li-Bewick-Shang-Fagan (2014)]

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$$r_1 = g(-\infty) < 0 < g(+\infty) = r_2$$

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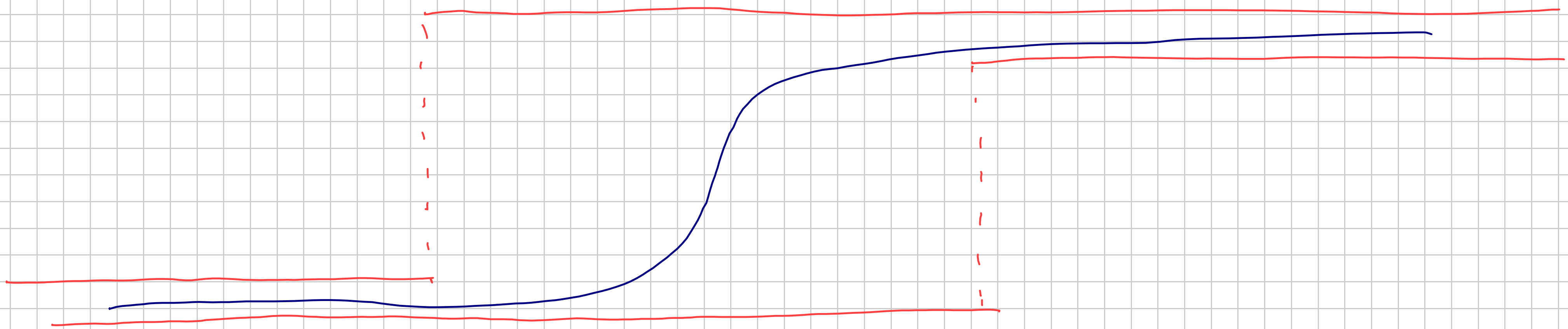
We consider the special case $r(x,t) = \begin{cases} r_2 & x \geq c_1 t \\ r_1 & x < c_1 t \end{cases}$ $\quad r_1 < 0 < r_2$

In general, $r(x,t) = g(x - c_1 t)$ can be approximated by similar step functions.



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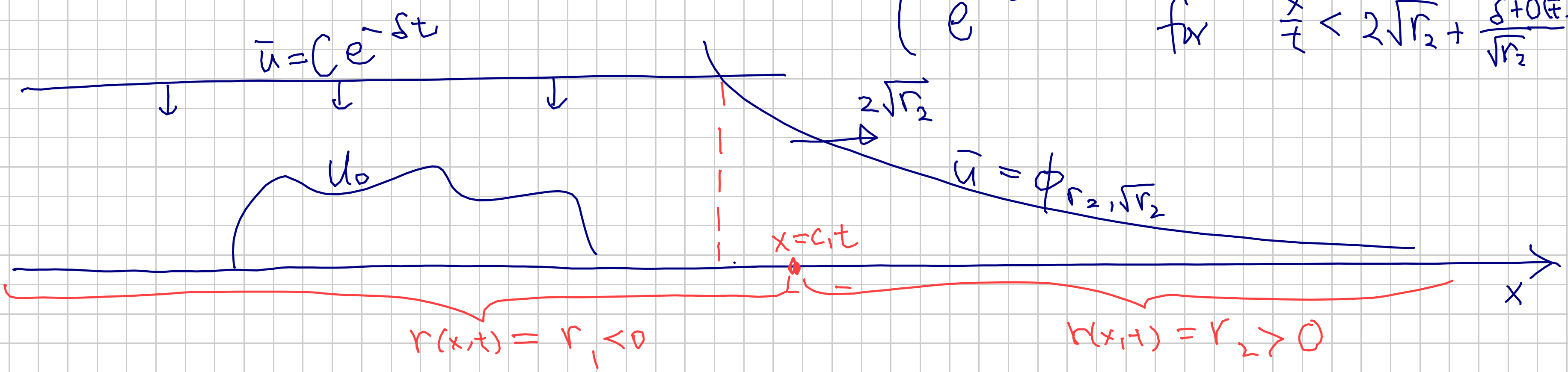
In general, $r(x,t) = g(x - c_1 t)$ can be approximated by similar step functions.

**Pf of (a)** Construct the supersolution. Let $\delta = \min\{\sqrt{r_2}\,(c_1 - 2\sqrt{r_2}),\ -\frac{r_1}{2}\}$.

$$\bar{u}(x,t) = \min\{e^{-\sqrt{r_2}(x - 2\sqrt{r_2}\,t)},\ Ce^{-\delta t}\} = \begin{cases} e^{-\sqrt{r_2}(x - 2\sqrt{r_2}\,t)} & \text{for } \frac{x}{t} \geq 2\sqrt{r_2} + \frac{\delta + O(\frac{1}{t})}{\sqrt{r_2}} \\ e^{-\delta t} & \text{for } \frac{x}{t} < 2\sqrt{r_2} + \frac{\delta + O(\frac{1}{t})}{\sqrt{r_2}} \end{cases}$$



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(b) $0 < c_1 < 2\sqrt{r_2}$ $\Longrightarrow$ (persistence + spreading at $2\sqrt{r_2}$)
$$\begin{cases} \displaystyle\lim_{t\to\infty} \sup_{x > (2\sqrt{r_2}+\varepsilon)t} u(x,t) = 0 & \forall\, \varepsilon > 0 \\ \displaystyle\liminf_{t\to\infty} \inf_{(c_1+\varepsilon)t < x < (2\sqrt{r_2}-\varepsilon)t} u(x,t) > 0 & \forall\, 0 < \varepsilon \ll 1. \end{cases}$$

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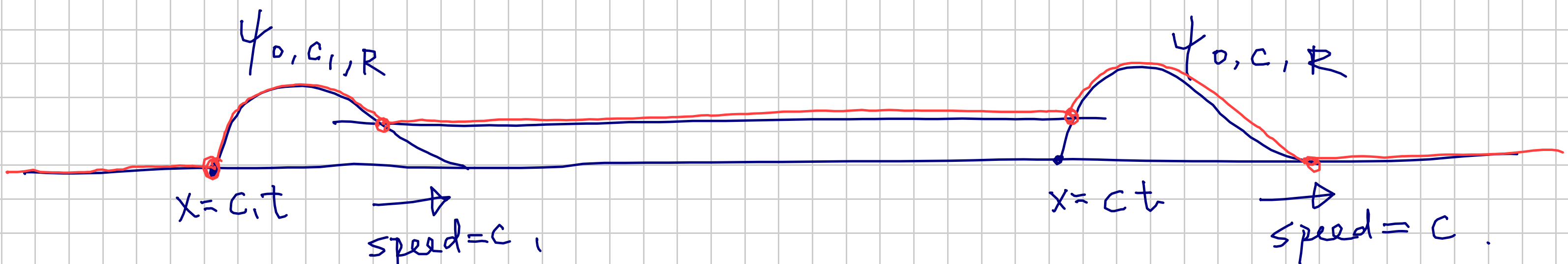
Pf of (b) is similar as in $r(x,t) \equiv r_0$. Let $0 < c_1 < 2\sqrt{r_2}$

Fix $c \in (c_1, 2\sqrt{r_2})$, and $R \gg 1$, define $\psi(x,t)$ by




Then for $\eta > 0$ small, $\underline{u}(x,t) = \eta\,\psi(x,t)$ is subsolution.

We omit the details. #

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$$g(-\infty) = r_1 < 0 < r_2 = g(+\infty)$$

(a) $\quad c_1 > 2\sqrt{r_2} \implies$ (extinction) $\quad \lim_{t\to\infty} \sup_{x\in\mathbb{R}} u(x,t) = 0$.

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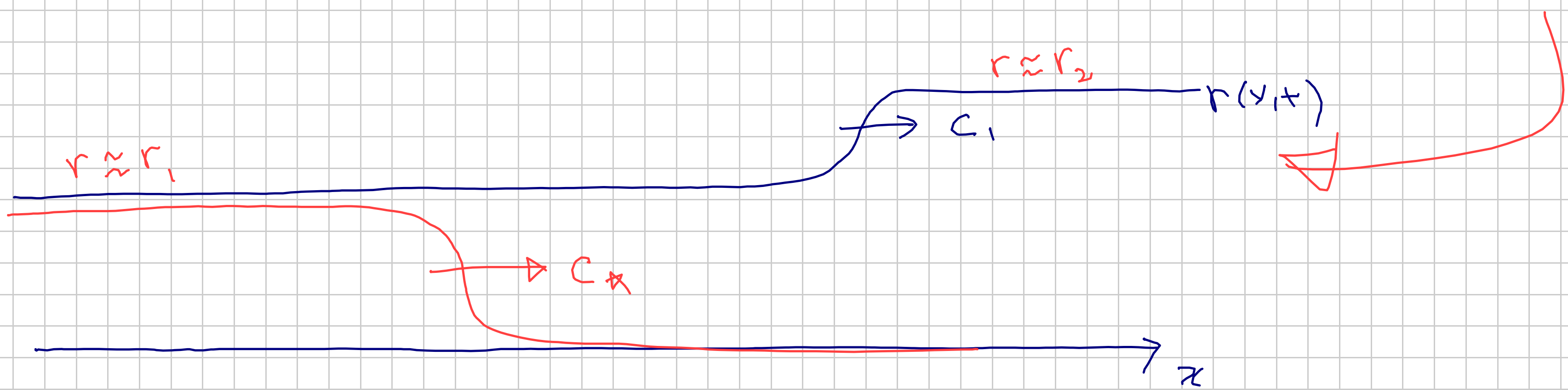
See also [T. Yi & Zhao (2020)] for generalization to monotone dynamical systems "sandwiched" between two limiting homogeneous KPP systems, where one is "positive" and the other one is "negative".

## § Shifting positive environment and nonlocal pulling

Consider the case $r(x,t) = g(x - c_1 t)$, where $g$ is increasing, and $\boxed{0 < r_1 < r_2, \quad \text{where} \quad r_1 = g(-\infty), \quad r_2 = g(+\infty).}$

- $c_* \in [2\sqrt{r_1}, 2\sqrt{r_2}]$
- $c_1 = 0$ or small $\longrightarrow$ $c_* = 2\sqrt{r_2}$
- $c_1 \gg 1$ $\longrightarrow$ $c_* = 2\sqrt{r_1}$



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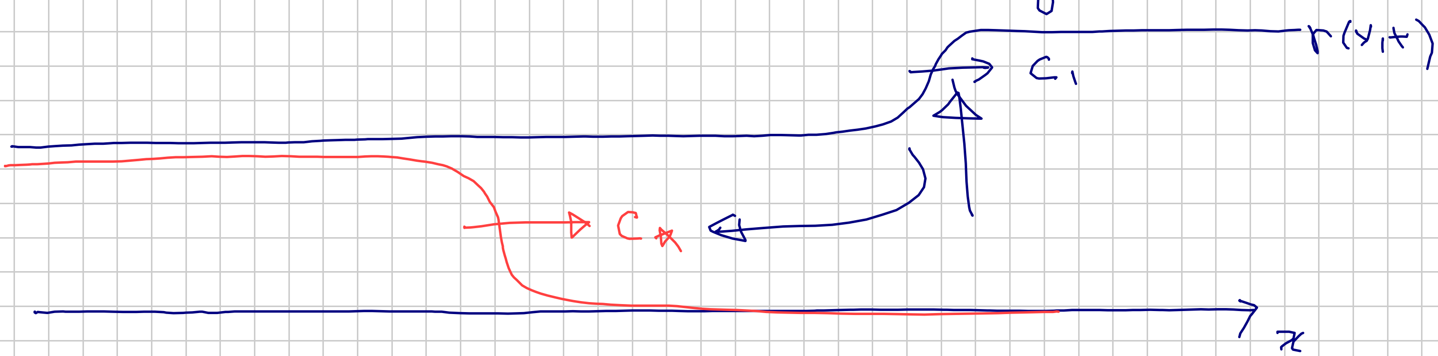
**Thm C** [L.-Yu (2021)]

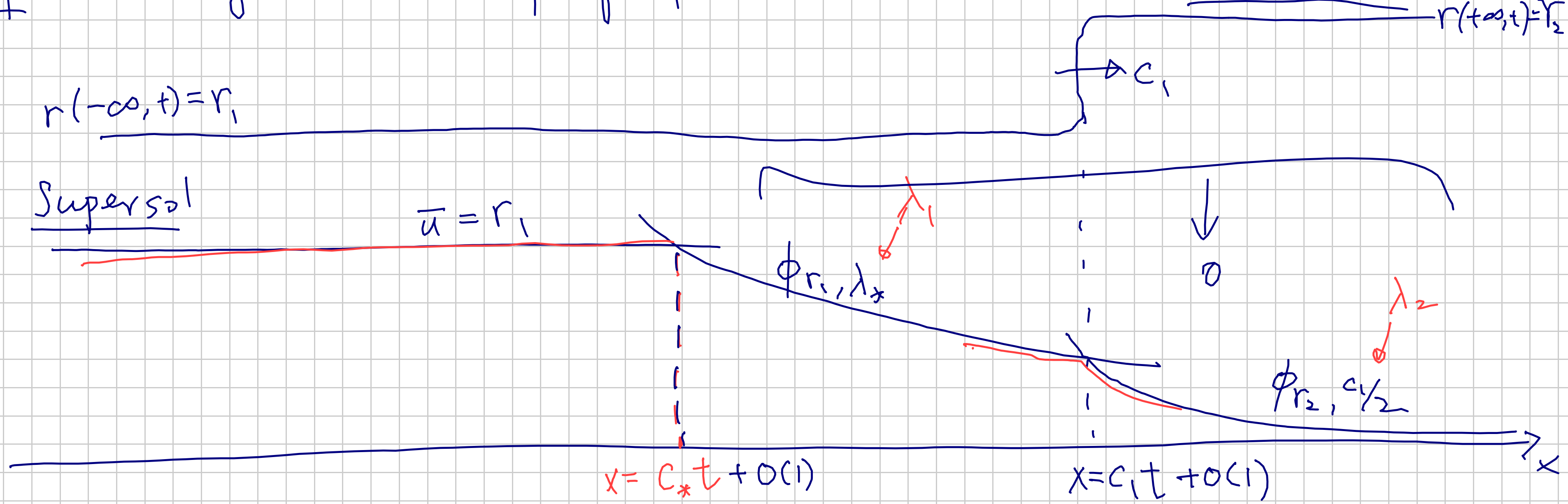
$$c_* = \begin{cases} 2\sqrt{r_2} & \text{if} \quad 0 \le c_1 \le 2\sqrt{r_2} \\ \frac{c_1}{2} - \sqrt{r_2 - r_2} + \dfrac{r_1}{\frac{c_1}{2} - \sqrt{r_2 - r_1}} & \text{if} \quad 2\sqrt{r_2} < c_1 < 2\left(\sqrt{r_2 - r_1} + \sqrt{r_1}\right) \\ 2\sqrt{r_1} & \text{if} \quad c_2 \ge 2\left(\sqrt{r_2 - r_1} + \sqrt{r_1}\right). \end{cases}$$

In cases 1 and 3, $c_*$ is determined by the environment where the transition front is located. $\to$ locally-pulled

In case 2, $c_*$ is influenced by the speed of the shifting env.



Pf We only sketch the proof of the case $2\sqrt{r_2} < c_1 < 2\left(\sqrt{r_2-r_1}+\sqrt{r_1}\right)$




where $c_* = \lambda_* + \dfrac{r_1}{\lambda_*}$ and $\lambda_* = \dfrac{c_1}{2} - \sqrt{r_2 - r_1} \in (0, \sqrt{r_1})$

check at $x = c_1 t$,
$$\begin{cases} 0 > \partial_x \phi_{r_1,\lambda_*} \geq \partial_x \phi_{r_2,c_1/2} \\ \phi_{r_1,\lambda_*} = \phi_{r_2,c_1/2} \end{cases} \quad \text{(exercise)}.$$

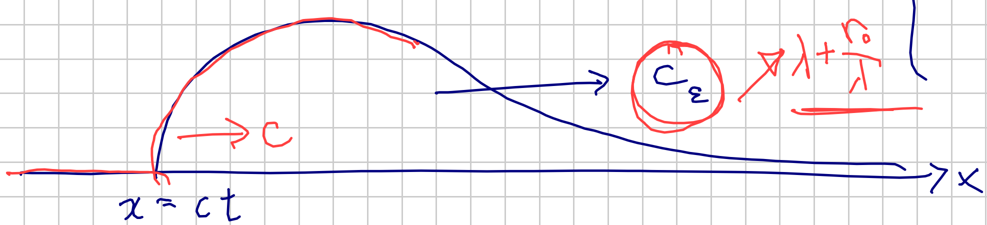
Remark $(\lambda_1, \lambda_2) = (\lambda_*, c_1/2)$ is "optimized" among all "admissible" choices of $\lambda_1 \in (0, \sqrt{r_1}]$ and $\lambda_2 \in (0, \sqrt{r_2}]$ to produce the slowest possible supersol. $\bar{u}(x,t)$.

**Subsol** Given $\lambda \in (0, \sqrt{r_0})$ and any $\varepsilon > 0$ suff small such that

$$\lambda < \sqrt{r_0 - \varepsilon} \quad \text{and} \quad 0 < \varepsilon < \frac{r_0 - \varepsilon}{\lambda} - \lambda$$

Let $C = C_\varepsilon = \lambda + \frac{r_0 - \varepsilon}{\lambda}$, then $\varepsilon < C - 2\lambda$.

Define $\rho(x,t) = \rho_{r_0, \lambda, \varepsilon}(x,t) = \begin{cases} e^{-\lambda(x-ct)} - e^{-(\lambda+\varepsilon)(x-ct)} & \text{for } x - ct > 0 \\ 0 & \text{for } x - ct \le 0. \end{cases}$




Note that $\sup \rho \le 1$.

**Claim** For $0 < \eta \le \varepsilon$, $(\eta\rho)_t - (\eta\rho)_{xx} - (\eta\rho)(r_0 - \eta\rho) \le 0$, in gen. sense.

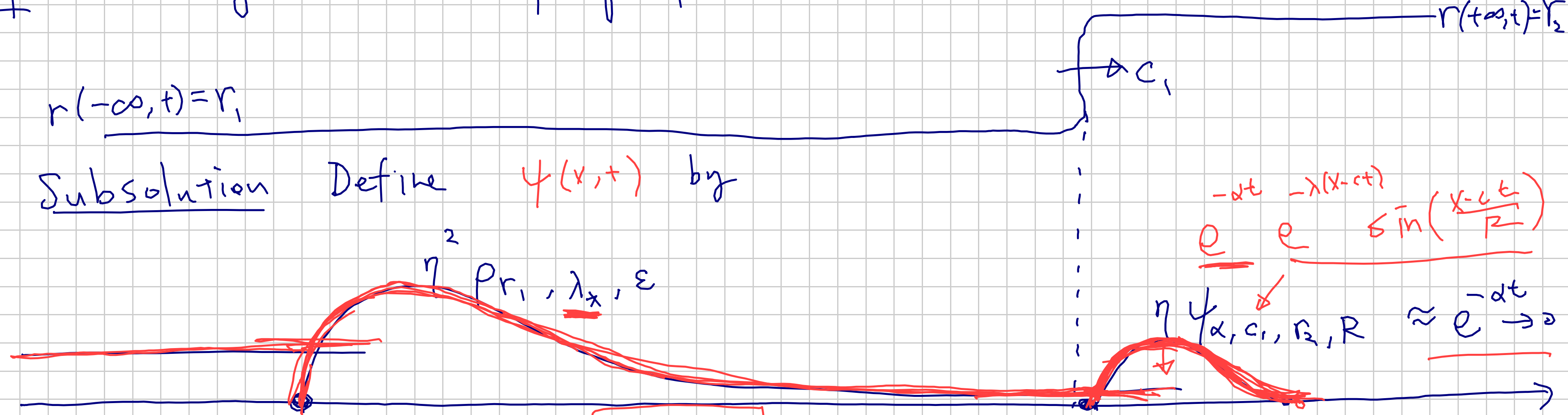
**Check** $\rho_t - \rho_{xx} - (r_0 - \varepsilon)\rho$

$$= e^{-\lambda(x-ct)}\left[c\lambda - \lambda^2 - r_0 + \varepsilon\right] - e^{-(\lambda+\varepsilon)(x-ct)}\left[c(\lambda+\varepsilon) - (\lambda+\varepsilon)^2 - r_0 + \varepsilon\right]$$

$$= -e^{-(\lambda+\varepsilon)(x-ct)}\left[c\varepsilon - 2\lambda\varepsilon - \varepsilon^2\right] = -e^{-(\lambda+\varepsilon)(x-ct)}\varepsilon\left[c - 2\lambda - \varepsilon\right] \le 0$$

$$(\eta\rho)_t - (\eta\rho)_{xx} - \eta\rho(r_0 - \eta\rho) \le \eta\rho(-\varepsilon + \eta\rho) \le \eta\rho(-\varepsilon + \eta) \le 0.$$

Pf We only sketch the proof of the case $2\sqrt{r_2} < c_1 < 2\left(\sqrt{r_2 - r_1} + \sqrt{r_1}\right)$




Subsolution Define $\psi(x,t)$ by $e^{-\alpha t}\, e^{-\lambda(x-ct)} \sin\left(\frac{x-ct}{R}\right)$

$x = c_\varepsilon t$ (where $c_\varepsilon = \boxed{\lambda_* + \frac{r_1 - \varepsilon}{\lambda_*}} \to \lambda_* + \frac{r_1}{\lambda_*}$

Fix ① $\lambda_* = \frac{c_1}{2} - \sqrt{r_2 - r_1} \in (0, \sqrt{r_1})$ and $\varepsilon > 0$ small $\Longrightarrow$ $\rho_{r_1,\lambda_*,\varepsilon}$ is subsol.

② $\rho_{r_1,\lambda_*,\varepsilon}\Big|_{x=c_1 t} = \exp\left(-\lambda_*(c_1 - c_\varepsilon)t\right)(1+o(1)) \to$ take $\alpha = \lambda_*(c_1 - c_\varepsilon)$.

③ Verify that $\alpha > -r_2 + \frac{c_1^2}{4}$ (exercise)

then choose $R \gg 1$ s.t. $\alpha > -r_2 + \frac{c_1^2}{4} + \frac{1}{R^2}$.

$\Rightarrow$ For $\varepsilon, \eta > 0$ small, $\eta^2 \rho_{r_1,\lambda_*,\varepsilon}$ and $\eta\, \psi_{\alpha,c_1,r_2,R}$ are subsol.

$\Rightarrow$ spreading speed $\geq c_\varepsilon$ $\forall \varepsilon > 0$

## § Shifting positive environment and nonlocal pulling

Consider the case $r(x,t) = g(x - c_1 t)$, where $g$ is increasing, and $0 < r_1 < r_2$, where $r_1 = g(-\infty)$, $r_2 = g(+\infty)$.

**Thm C**

$$c_* = \begin{cases} 2\sqrt{r_2} & \text{if } 0 \le c_1 \le 2\sqrt{r_2} \\ \dfrac{c_1}{2} - \sqrt{r_2 - r_2} + \dfrac{r_1}{\frac{c_1}{2} - \sqrt{r_2 - r_1}} & \text{if } 2\sqrt{r_2} < c_1 < 2\left(\sqrt{r_2 - r_1} + \sqrt{r_1}\right) \\ 2\sqrt{r_1} & \text{if } c_2 \ge 2\left(\sqrt{r_2 - r_1} + \sqrt{r_1}\right). \end{cases}$$

**Remarks**

1. The phenomenon was first found by [Holzer-Scheel SIMA 2014], by looking at the moving coordinate $y = x - c_1 t$ and using eigenvalue method.
2. The name "nonlocal-pulling" was introduced in [Girardin-Lam 2019], where it is treated by super/subsol method.
3. Using Hamilton-Jacobi equations, one can treat the general case: $r\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \xrightarrow{\varepsilon \to 0} R\left(\frac{x}{t}\right)$ for any piecewise constant $R$. See [L-Yu] for precise conditions.
4. See also [Berestycki-Nadin, Mem. AMS (2022)].

<u>Remarks</u> 4. See [Faye-Giletti-Holzer, DCDS-s (2022)] for the case of shifting diffusivity.

5. In [Berestycki-Nadin, Mem. AMS (2022)], the Hamilton-Jacobi Method was combined with homogenization ideas to treat very general heterogeneity.

We close by briefly describing the Hamilton-Jacobi Approach.

Consider $u^{\varepsilon}(y,t) = \underline{u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)}$ and $r^{\varepsilon}(y,t) = r\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \to \begin{cases} r_2 & \frac{x}{t} > c_1 \\ r_1 & \frac{x}{t} < c_1 \end{cases}$

Then
$$\begin{cases} \partial_t u^{\varepsilon} - \varepsilon \partial_{xx} u^{\varepsilon} = \frac{1}{\varepsilon} u^{\varepsilon}(r^{\varepsilon} - u^{\varepsilon}) & x \in \mathbb{R},\ t > 0 \\ u^{\varepsilon}(x,0) = u_0\left(\frac{x}{\varepsilon}\right) & x \in \mathbb{R} \end{cases}$$


**WKB-Ansatz** $\boxed{w^{\varepsilon}(y,t) = -\varepsilon \log u^{\varepsilon}(x,t) \iff u^{\varepsilon}(x,t) = \exp\left(-\frac{w^{\varepsilon}(x,t)}{\varepsilon}\right).}$

then
$$\begin{cases} \partial_t w^{\varepsilon} - \varepsilon \partial_{xx} w^{\varepsilon} + |\partial_x w^{\varepsilon}|^2 + r^{\varepsilon} - u^{\varepsilon} = 0 & x \in \mathbb{R},\ t > 0 \\ w(x,0) = \begin{cases} 0 & x < 0 \\ \infty & x > 0 \end{cases} \end{cases}$$

Suppose $w^{\varepsilon}(t,x) \longrightarrow w(t,x)$ locally uniformly,

then $w$ can be determed as follows:

$$w(x,t) = \max\{J(x,t), 0\},$$ where $J$ is the unique viscosity sol of

$$\begin{cases} \partial_t J + |\partial_x J|^2 + r_2 - (r_2 - r_1)\chi_{\{x \le c_1 t\}} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ J(x,0) = \begin{cases} 0 & x < 0 \\ \infty & x > 0 \end{cases} \end{cases}$$

## WKB-Ansatz

$$w^\varepsilon(x,t) = -\varepsilon \log u^\varepsilon(x,t) \iff u^\varepsilon(x,t) = \exp\left(-\frac{w^\varepsilon(x,t)}{\varepsilon}\right).$$

$w(t,x) = \lim_{\varepsilon\to 0} w^\varepsilon(x,t)$ can be determed as follows:

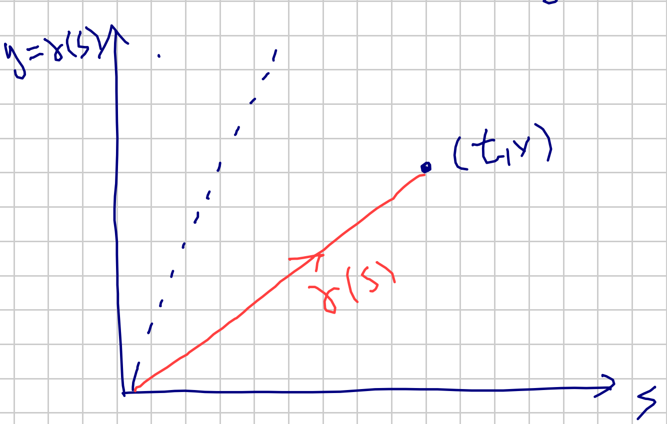
$\boxed{w(x,t) = \max\{J(x,t), 0\}}$, where $J$ is the unique viscosity sol of

$$\begin{cases} \partial_t J + |\partial_x J|^2 + r_2 - (r_2 - r_1)\chi_{\{x \le c_1 t\}} = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ J(x,0) = \begin{cases} 0 & x > 0 \\ \infty & x > 0 \end{cases} \end{cases}$$

By [Evans-Souganidis (1989)], $J(x,t)$ can be determind by an optimization prob.

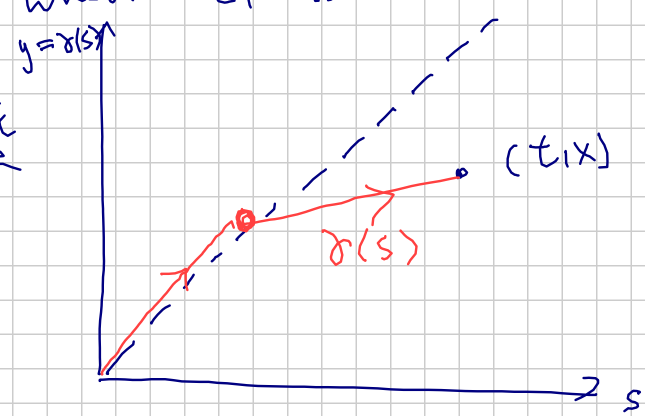
$$J(x,t) = \inf_{\substack{\gamma \in W^{1,\infty}([0,t]) \\ \gamma(t) = x,\ \gamma(0)=0}} \left\{ \int_0^t \frac{|\dot\gamma(s)|^2}{4} - r_2 + (r_2 - r_1)\chi_{\{\gamma(s) \le c_1 s\}}\, ds \right\}$$

when $c_1$ is too large




Suppose $(x,t)$ is at the front

when $c_1$ is not too large



In fact,

$$J(x,t) = \begin{cases} \frac{t}{4}\left(\frac{x^2}{t^2} - 4r_2\right) & \text{for } \frac{x}{t} \geq c_1 \\ \left(\frac{c_1}{2} - \sqrt{r_2 - r_1}\right)(x - c_* t) & \text{for } c_1 - 2\sqrt{r_2 - r_1} \leq \frac{x}{t} < c_1 \\ \frac{t}{4}\left(\frac{x^2}{t^2} - 4r_1\right) & \text{for } 0 \leq \frac{x}{t} < c_1 - 2\sqrt{r_2 - r_1} \\ -tr_1 & \end{cases}$$

See [Lin-Lin-L. DCDS-A (2020)] for explicit calculations.

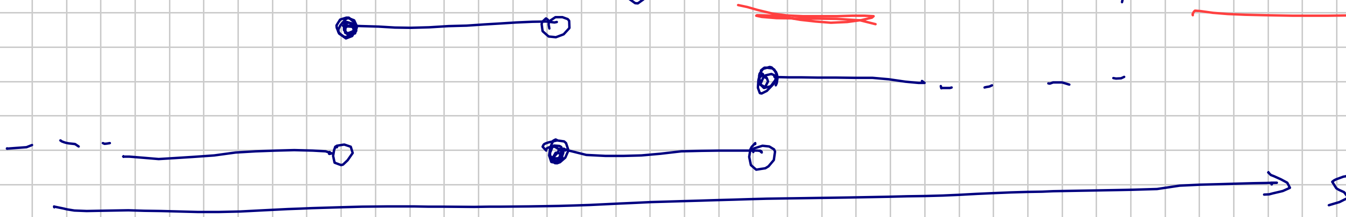
$\Rightarrow$ spreading speed is one of $2\sqrt{r_2}$, $2\sqrt{r_1}$ or $c_*$.

- More recently, [Berestycki-Nadin, Mem. AMS, in press] applied the HJ approach, where the Hamiltonian is defined by homogenization ideas, to treat very general heterogeneous environments.

- In [L.-Yu (2021), submitted], we used the theory of HJ with discontinuous Hamiltonian to treat general shifting environments:

  where $r\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \longrightarrow R(x/t)$,

  with $s \longmapsto R(s)$ being monotone, or piecewise constant....



- For locked waves, see [Berestycki-Fang, J.D.E. (2018)].
- Entire solutions?